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Two component Bose Einstein condensates: coexistence, segregation and vortex patterns

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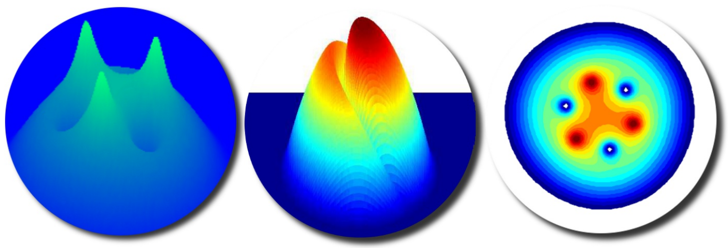
Joint works with

B. Noris and C. Sourdis (CMP 2014)
J. Royo-Letelier (Calculus of Variations and PDE's 2014)



Motivated from numerical simulations in

Aftalion-Mason (PRA 2012 and PRA 2013).

A two component Bose Einstein condensate is a mixture of 2 species describing:

2 different isotopes of the same alkali atom,
or isotopes of different atoms,
or a single isotope in 2 different hyperfine spin states.

Described by 2 wave functions ψ_1 and ψ_2 with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$ minimizing a Gross Pitaevskii energy with a coupling term.

The coupling can be either through the modulus or through the phase (spin orbit coupling or Rabi coupling).

Gross Pitaevskii energy for a single condensate

A single Bose Einstein condensate is in a state which minimizes

$$E(\psi) = \int_{\mathrm{R}^2} \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2\varepsilon^2} r^2 |\psi|^2 + \frac{g}{2\varepsilon^2}|\psi|^4,$$

under $\int |\psi|^2 = 1$. Mathematical limit: $\varepsilon \to 0$.

$$-\frac{\varepsilon^2}{2}\Delta\psi + \frac{1}{2}r^2\psi + |\psi|^2\psi = \lambda\psi$$

Leading order, inverted parabola profile:
$|\psi|^2 = \lambda^2 - (1/2)r^2$. Exponential decay at infinity.

Ignat-Millot Uniqueness and convergence to Thomas Fermi profile

Karali-Sourdis very precise estimates. Painlevé boundary layer. See also Gallo-Pelinovski. Based on perturbation arguments to construct an approximate solution, and then use the properties of the linearized operator to get a true solution. The uniqueness implies that it is the ground state.

Two component condensates

2 wave functions ψ_1 and ψ_2 with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$

$$E_g(\psi) = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2\varepsilon^2}r^2|\psi|^2 + \frac{g}{2\varepsilon^2}|\psi|^4,$$

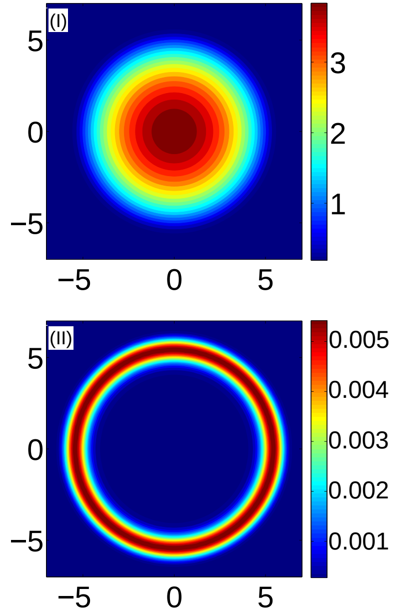
$$E = E_{g_1}(\psi_1) + E_{g_2}(\psi_2) + \frac{g_{12}}{\varepsilon^2}\int |\psi_1|^2|\psi_2|^2$$

ε: small parameter

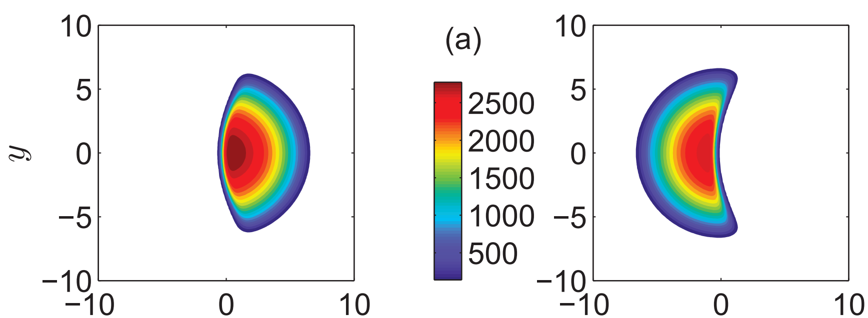
Important parameter $\Gamma_{12} = 1 - \frac{g_{12}^2}{g_1 g_2}$.

$\Gamma_{12} > 0$: coexistence of the components

$\Gamma_{12} < 0$: segregation (breaking of symmetry)



Coexistence case: 2 disks or a disk and an annulus



Segregation case

We look for positive solutions of

$$-\frac{1}{2}\Delta\psi_1 + \frac{\psi_1}{\varepsilon^2}(g_1|\psi_1|^2 + g_{12}|\psi_2|^2) = \frac{1}{\varepsilon^2}\psi_1(\lambda_1 - \frac{1}{2}r^2)$$

$$-\frac{1}{2}\Delta\psi_2 + \frac{\psi_2}{\varepsilon^2}(g_{12}|\psi_1|^2 + g_2|\psi_2|^2) = \frac{1}{\varepsilon^2}\psi_2(\lambda_2 - \frac{1}{2}r^2)$$

with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$. Thomas-Fermi profile given by

$$g_1|\psi_1|^2 + g_{12}|\psi_2|^2 = \lambda_1 - \frac{1}{2}(1-\Omega^2)r^2$$

$$g_{12}|\psi_1|^2 + g_2|\psi_2|^2 = \lambda_2 - \frac{1}{2}(1-\Omega^2)r^2$$

In the case $\Gamma_{12} > 0$, there is a solution to the reduced limiting system, there is a unique positive solution and we can analyze the convergence.

In the case $\Gamma_{12} < 0$, there is a Γ convergence to an interface problem.

Coexistence case, $\Gamma_{12} > 0$

leading order, inverted parabola profile:

$$g_1|\psi_1|^2 + g_{12}|\psi_2|^2 = \lambda_1 - \frac{1}{2}(1-\Omega^2)r^2$$

$$g_{12}|\psi_1|^2 + g_2|\psi_2|^2 = \lambda_2 - \frac{1}{2}(1-\Omega^2)r^2$$

Either 2 disks with different radii if $g_{12} < \frac{g_1+\sqrt{g_1^2+8g_1g_2}}{4}$ (if $g_1 \leq g_2$), or a disk and an annulus. Convergence in the TF limit. Aftalion-Noris-Sourdis following Aftalion-Jerrard-Letelier and Karali-Sourdis:

- uniqueness of the ground state and of solutions with decay. We use the division trick of Lassoued-Mironescu to prove existence. No moving plane method works to get radial symmetry.

- precise estimate of the convergence to the Thomas-Fermi limit. Proved by constructing an approximate solution. Then using the linearized operator, we perturb it to a genuine solution. By uniqueness, it is the ground state.

Inside the Thomas-Fermi radius: convergence in $\varepsilon^2|\log\varepsilon|$.

Outside: exponentially small.

Size of $\varepsilon^{1/3}$ around the Thomas-Fermi radius.

As in Aftalion, Jerrard, Royo-Letelier, we can prove that the solution remains real valued (no vortex in the small density region) until the critical rotational value for nucleation of the 1st vortex in the bulk.

Segregation case

If $\Gamma_{12} = 1 - \frac{g_{12}^2}{g_1 g_2} < 0$, phase separation is expected: asymptotic limit $\Gamma_{12} \to -\infty$, or $g_{12} \to \infty$. The coexistence region gets asymptotically small. Two droplets are expected.

We define $\rho_T = |\psi_1|^2 + |\psi_2|^2$, $\psi_k = \sqrt{\rho_T}\chi_k$, $\chi_k = |\chi_k| e^{i\theta_k}$ so that $|\chi_1|^2 + |\chi_2|^2 = 1$ and $S_z = |\chi_1|^2 - |\chi_2|^2$. We have $S_z = 1$ when only component 1 is present, $S_z = -1$, when only component 2 is present.

- $\Gamma_{12} \to -\infty$, $g_1 = g_2$: Thomas Fermi regime with inverted parabola profile for $\rho_T = |\psi_1|^2 + |\psi_2|^2$. Gamma convergence to a De Giorgi type problem (Aftalion- Royo-Letelier). Write $S_z = \cos\phi$, then the energy becomes

$$\int |\nabla\sqrt{\rho_T}|^2 + \frac{\rho_T}{2}|\nabla\phi|^2 + \frac{1}{2\varepsilon^2}r^2\rho_T + g_{12}\frac{\rho_T^2}{4\varepsilon^2}(1-\cos^2\phi) + g_1\frac{\rho_T^2}{4\varepsilon^2}(1+\cos^2\phi)$$

If g_{12} is large, then $\cos^2\phi \sim 1$ almost everywhere, except on a boundary layer.
ρ_T is almost TF, and vanishes at interface.

We go back to the GP energy for a single condensate:

$$E_\varepsilon(\eta) = \int \frac{1}{2}|\nabla\eta|^2 + \frac{1}{2}r^2|\eta|^2 + \frac{g_1}{2\varepsilon^2}|\eta|^4.$$

under $\int \eta^2 = N_1 + N_2$. We call η the ground state. Let $\rho_T = \eta v$. Then the energy splits into

$$E_\varepsilon(\eta) + F_\varepsilon(v) + G_\varepsilon(\phi)$$

with

$$F_\varepsilon(v) = \int \frac{1}{2}\eta^2|\nabla v|^2 + \frac{g_1}{2\varepsilon^2}\eta^4(1-|v|^2)^2$$

$$G_\varepsilon(\phi) = \int \frac{1}{2}\eta^2 v^2|\nabla\phi|^2 + \frac{g}{2\varepsilon^2}(1-\frac{g_1}{g})\eta^4 v^4(1-\cos^2\phi)$$

F_ε is a Modica Mortola type energy with weight.

$|v|$ is 1 almost everywhere, but goes to zero on the interface region between the two components. Note that $g \to \infty$, so that $\varepsilon/\sqrt{g} << \varepsilon$.

We prove that G_ε converges to 0 and F_ε converges to $c_* \int_{interface} \eta^3$.

Slicing device of Ambrosio-Tortorelli

Limiting problem

defined by the inverted parabola $\eta^2 = (\lambda - \frac{1}{2}r^2)_+$, where D is the disk of radius $\sqrt{\lambda}/2$ and $\int_D \eta^2 = N_1 + N_2$.

Find the optimal D_1 and D_2 such that

$D = D_1 \cup D_2$, $\int_{D_1} \eta^2 = N_1$, $\int_{D_2} \eta^2 = N_2$ and they minimize

$\int_{\partial D_1 \cap \partial D_2} \eta^3$.

Better to have half spaces than disk+annulus to minimize this interface integral, if N_1, N_2 are not too small.

In the case of no trapping, results about the limiting problem: Sternberg-Zumbrum and recently Alikakos-Faliagas.

Related results of Berestycki-Lin-Wei-Zhao (no trapping potential). See also for bounded domains Caffarelli-Lin, Dancer et al, Noris-Tavares-Terracini-Verzini.

If g is of order 1, then the 2 functionals interplay in the limit and v goes down to m instead of 0 (recent result of Goldman, Royo-Letelier).

If $g_1 \neq g_2$, then the limiting problem is on the Thomas Fermi profile: the two radii are not the same.

Spin orbit coupling

Spin orbit coupled condensates

$$\int \sum_{k=1,2} \left(\frac{1}{2}|\nabla \psi_k|^2 + \frac{1}{2} r^2 |\psi_k|^2 + \frac{g_k}{2}|\psi_k|^4 \right) + g_{12}|\psi_1|^2|\psi_2|^2$$

$$-\kappa \psi_1^* \left(i\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_2}{\partial y} \right) - \kappa \psi_2^* \left(i\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_1}{\partial y} \right)$$

under the constraint $\int |\psi_1|^2 + |\psi_2|^2 = 1$.

We assume $g_1 = g_2 = g$ and define $\delta = g_{12}/g$.

Aftalion-Mason, PRA 2013

We define $\rho_T = |\psi_1|^2 + |\psi_2|^2$, $\psi_k = \sqrt{\rho_T}\chi_k$, $\chi_k = |\chi_k|e^{i\theta_k}$ so that $|\chi_1|^2 + |\chi_2|^2 = 1$ and $S_z = |\chi_1|^2 - |\chi_2|^2$, $S_x = \chi_1^*\chi_2 + \chi_2^*\chi_1$, $S_y = -i(\chi_1^*\chi_2 - \chi_2^*\chi_1)$.

$\delta > 1$: segregation: at $\kappa = 0$, one component is empty. As κ increases, to a giant skyrmion (disk+ thin annulus circulation 1), to multiple annuli and eventually stripes.

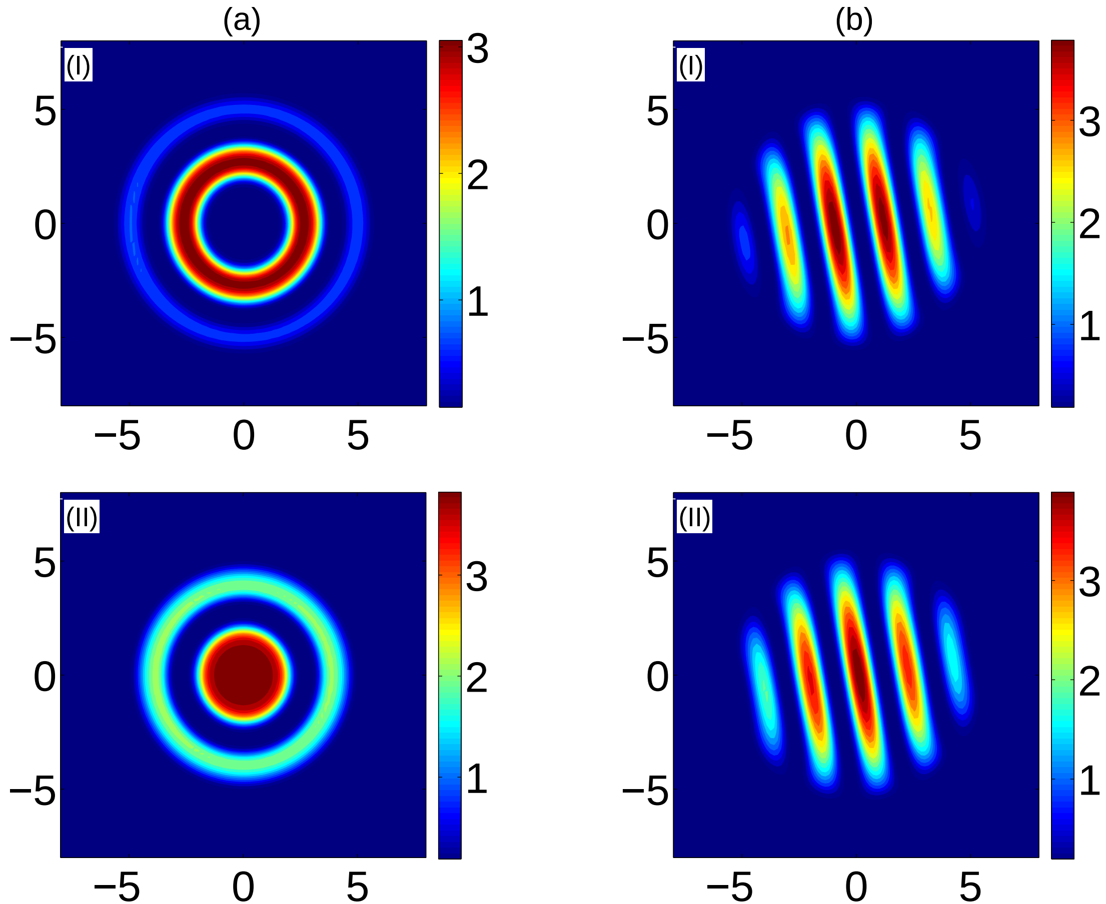


Figure 1: Left column (a): $(\delta, \kappa) = (1.5, 1.25)$ and right column (b): $(\delta, \kappa) = (1.5, 1.5)$. Density plots (frame (I), component-1, and (II), component-2).

Question: understand the Gamma limit of the spin orbit term in the segregation case?

$$-\kappa\psi_2^*\left(i\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_1}{\partial y}\right)$$

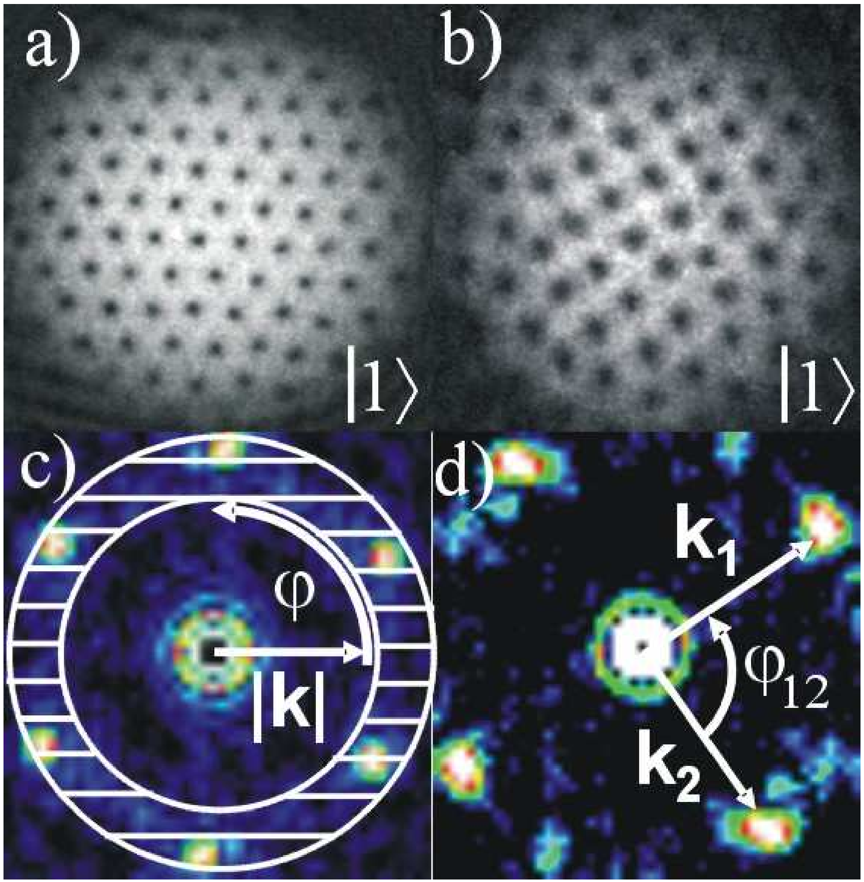
and how it competes with the term $\int_{interface}\eta^3$.

Formally in the case disk+annulus, we find that the circulation in each annulus is 1.

Two component condensates with rotation

2 component condensate: 2 wave functions, new phases and defects.

V. Schweikhard, I. Coddington, P. Engels, S. Tung, and E. A. Cornell (2004): a square lattice is stabilized in a two component condensate.



Two component condensates with rotation

(Aftalion-Mason) 2 wave functions ψ_1 and ψ_2 with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$

$$E_{\Omega,g}(\psi) = \int \frac{1}{2}|\nabla\psi - i\Omega \times r\psi|^2 + \frac{1}{2}r^2|\psi|^2(1-\Omega^2) + \frac{1}{2}g|\psi|^4,$$

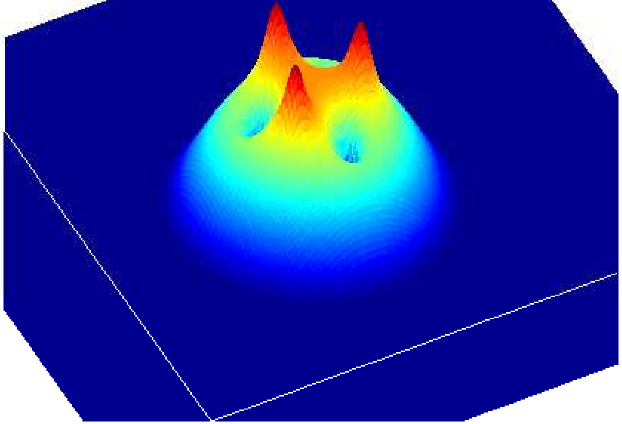
$$E = E_{\Omega,g_1}(\psi_1) + E_{\Omega,g_2}(\psi_2) + g_{12}\int |\psi_1|^2|\psi_2|^2$$

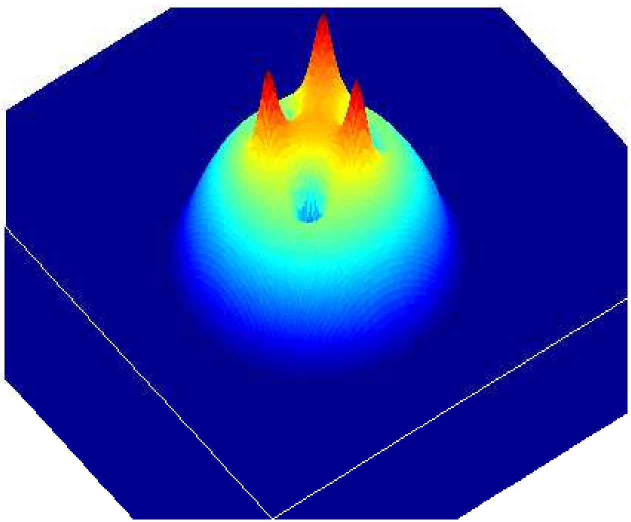
- g_{12} small: 2 components are disk-shaped with vortex lattices. a vortex in component 1 corresponds to a peak in component 2. Square lattice.

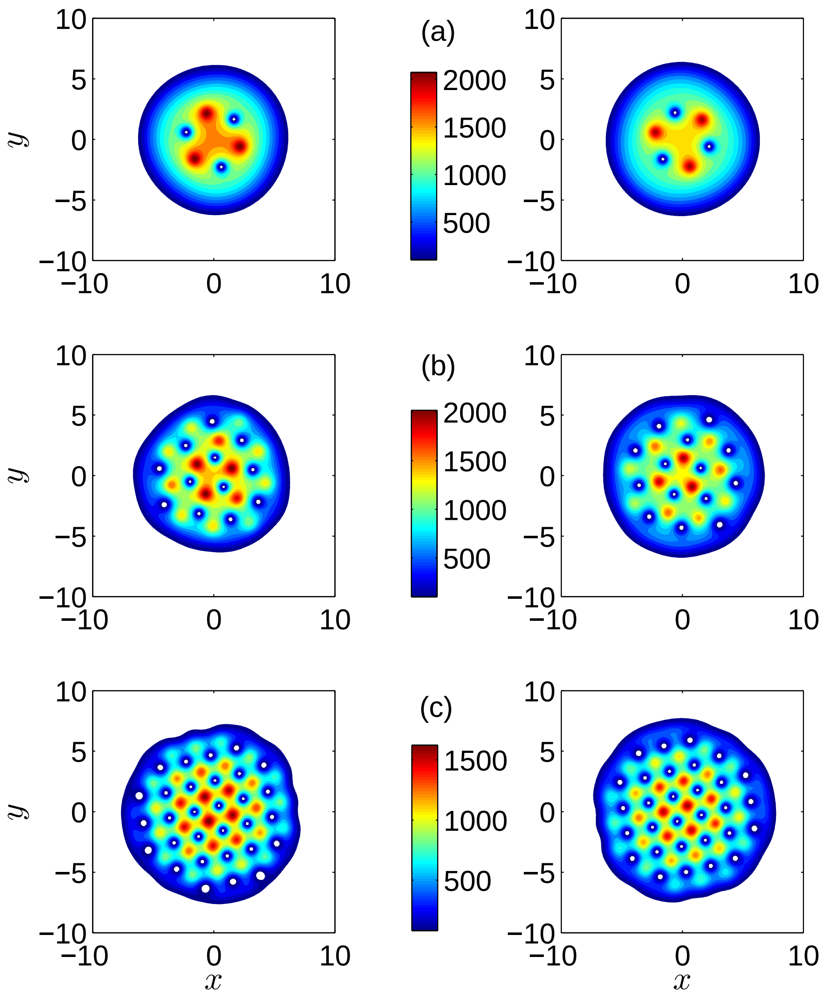
- g_{12} large: phase separation and breaking of symmetry: rotating droplets

- intermediate regime: phase separation but no breaking of symmetry, one component is a disk, the other is an annulus. Skyrmion in the boundary layer

- vortex sheets



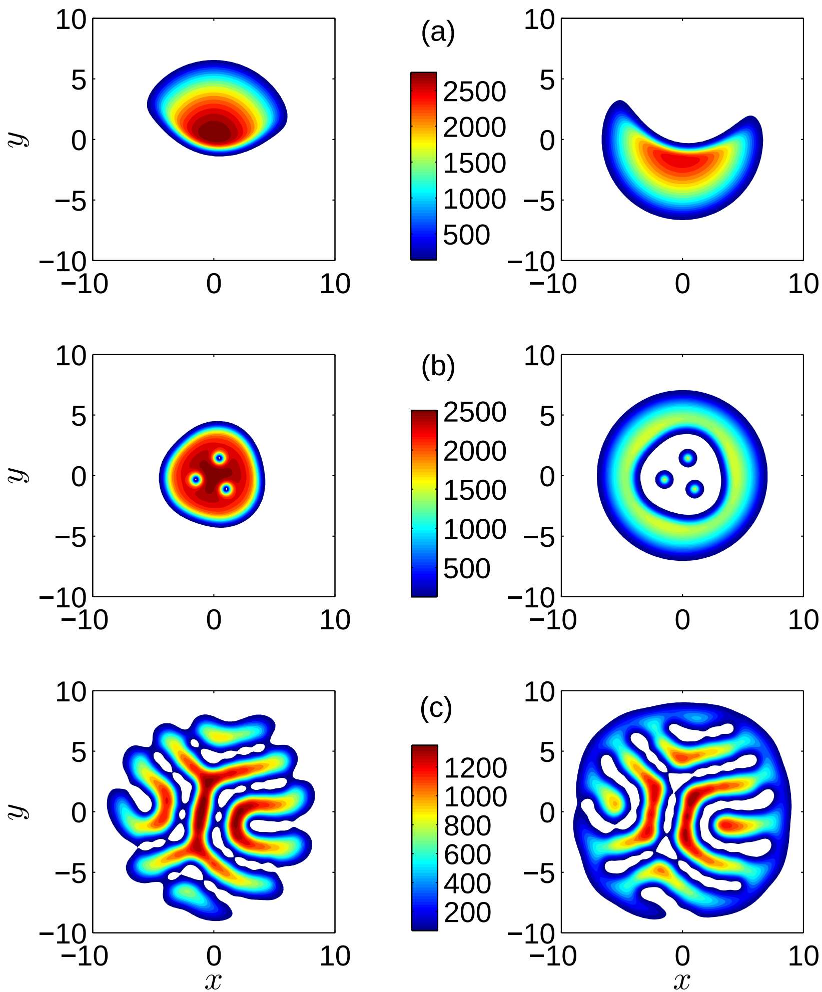




left column $|\psi_1|^2$
right column $|\psi_2|^2$

$\Omega =$ (a) 0.25, (b) 0.5, (c) 0.75

$g_1 = 0.0078$, $g_2 = 0.0083$, $N_1 = N_2 = 10^5$, $m_1 = m_2$, $g_{12} = 0.0057$

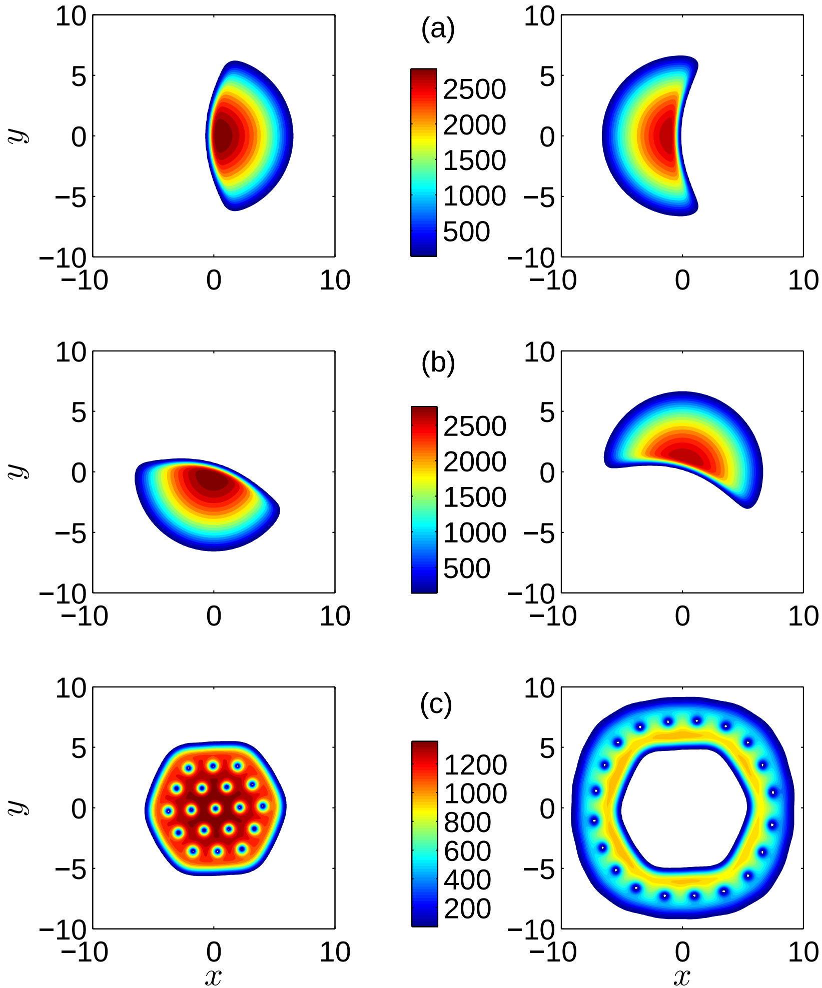


g_{12} large: phase separation

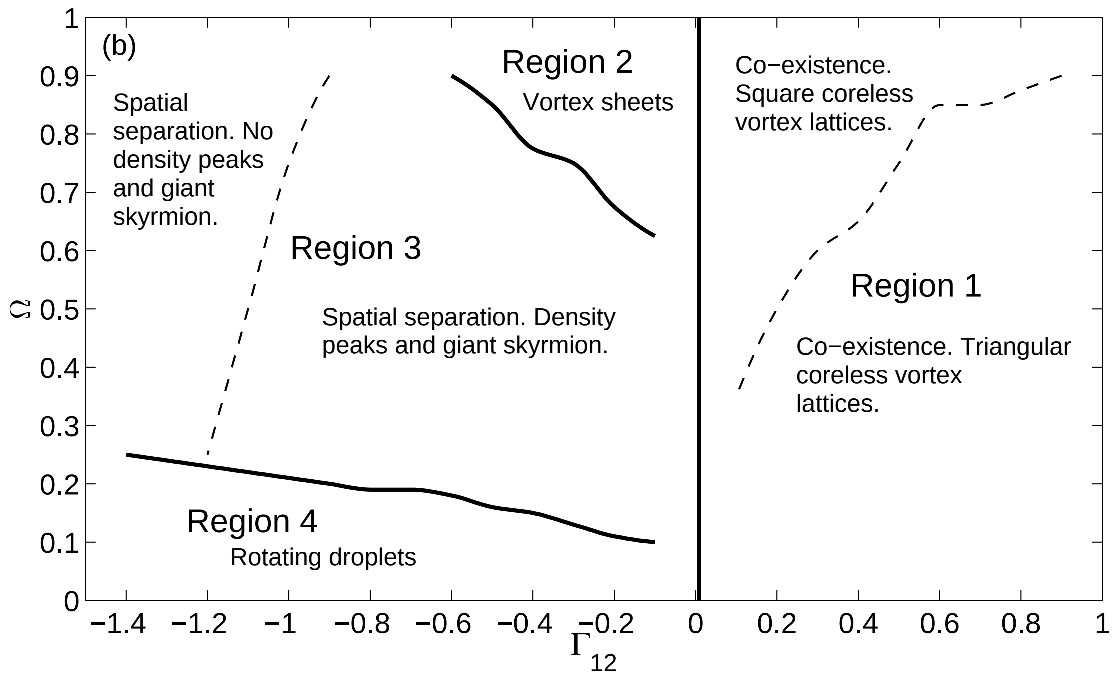
left column $|\psi_1|^2$

right column $|\psi_2|^2$

$g_1 = 0.0078$, $g_2 = 0.0083$, $N_1 = N_2 = 10^5$, $g_{12} = 0.0092$, $\Omega =$ (a) 0.1, (b) 0.5, (c) 0.9

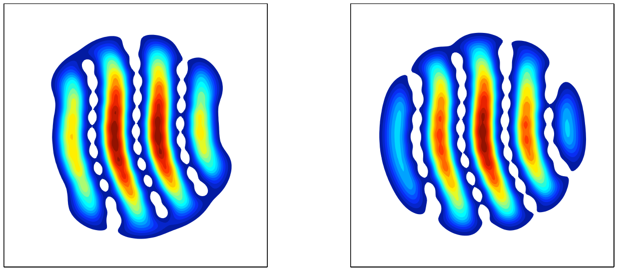


g_{12} larger
left column $|\psi_1|^2$
right column $|\psi_2|^2$
$g_1 = 0.0078$, $g_2 = 0.0083$, $N_1 = N_2 = 10^5$, $g_{12} = 0.0122$), $\Omega =$ (a) 0, (b) 0.1, and (c) 0.9.



$\Omega - \Gamma_{12}$ phase diagrams $g_1 = 0.0078$, $g_2 = 0.0083$, $N_1 = N_2 = 10^5$ $\Gamma_{12} = 1 - \frac{g_{12}^2}{g_1 g_2}$

Vortex sheets



Add rotation. This requires to understand the equation for S_z (or ϕ) at leading order.

Case with rotation

• until the first vortex, the minimizer is unique and real valued. Done by division of the ground state at Ω, by the ground state at $\Omega = 0$ and with jacobian estimates, we prove that the ratio is 1. It means that the ground state stays real valued until the first vortex. (Aftalion-Noris-Sourdis).

– computation of the critical velocity for the 1st vortex, called Ω_c (in component with larger radius). (Aftalion-Mason-Wei)

– vortex peak interaction. The equation of the vortex core has to be replaced by a system of vortex/spike $(f(r)e^{i\theta}, S(r))$ where $(f(r), S(r))$ satisfies

$$\frac{(rf')'}{r} - \frac{f}{r^2} + \alpha_1 f(1 - f^2) + \alpha_{12} S^2 f = 0,$$

$$\frac{(rS')'}{r} + \alpha_2 S(1 - S^2) + \alpha_{12} f^2 S = 0.$$

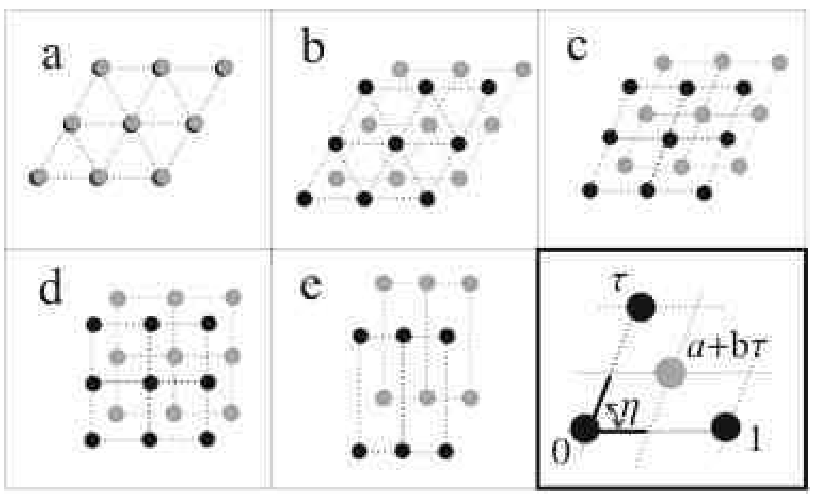
Related results by Eto, Kasamatsu, Nitta, Takeuchi, Tsubota, in the case of a homogeneous condensate.

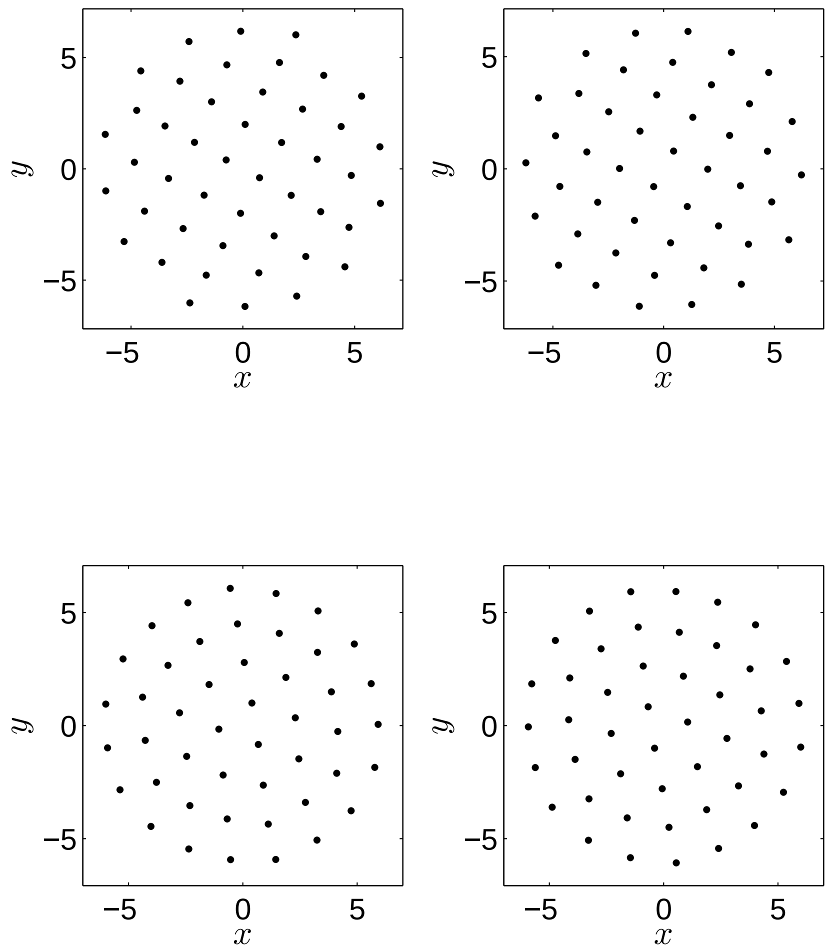
Existence of a non degenerate solution, upper bound for the full problem (Aftalion-Wei). Related results: Alama-Bronsard-Mironescu.

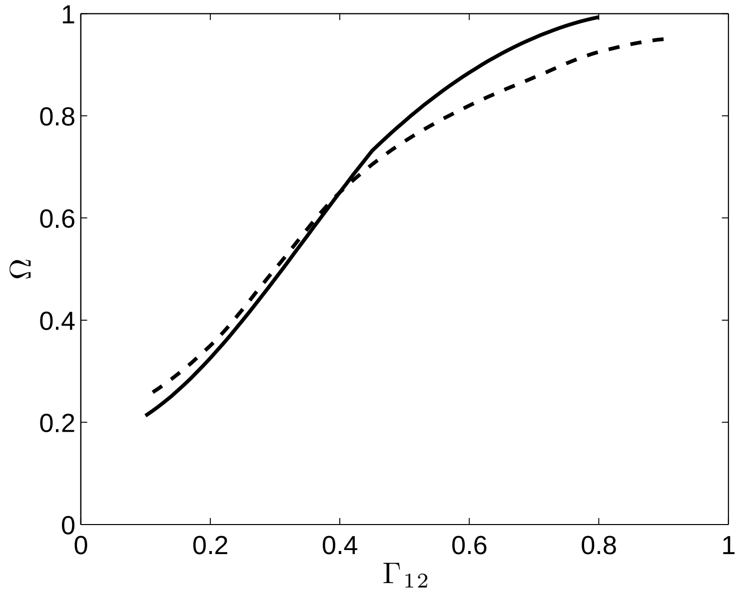
$$-\sum_{i,j}(\log|p_i - p_j| + \log|q_i - q_j|) + \sum_i(|p_i|^2 + |q_i|^2) + \sum_{i,j}\frac{c_\Omega}{|p_i - q_j|^2}$$

where p_i are the vortices for component 1, q_j are the vortices for component 2 and $c_\Omega = \frac{\pi(1-\Gamma_{12})|\log g_1|^2}{8\Gamma_{12}^2 N_1 g_1}(2\frac{\Omega}{\Omega_c} - 1)$. At some critical value of c_Ω, the lattice goes from triangular to square: relation between Γ_{12} and Ω.

From Kasamatsu-Tsubota-Ueda, vortex lattices:




y
x
−5
0
5

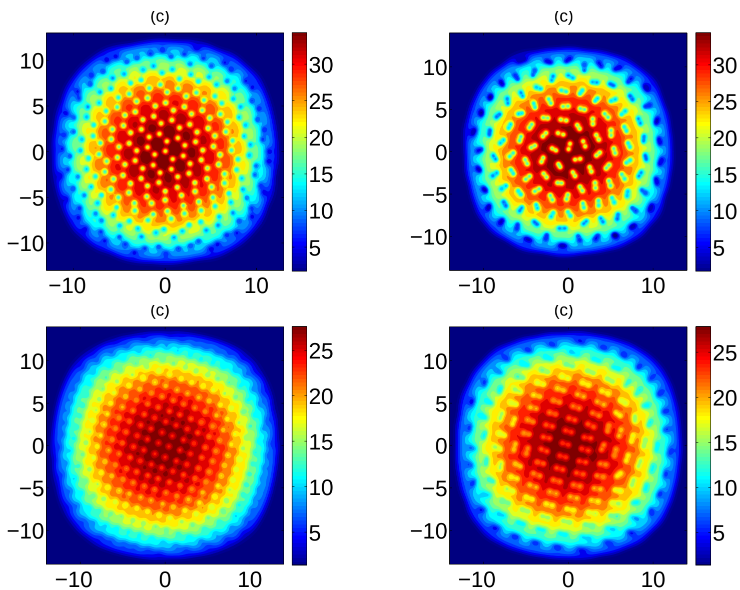

Ω
Γ12
0
0.2
0.4
0.6
0.8
1

Work in progress on Rabi coupling (with P.Mason)

The coupling is

$$\int \psi_1^* \psi_2 + \psi_1 \psi_2^*.$$

In the rotation case, different geometries on the vortex patterns for which we can get an asymptotic analysis



Open Questions

In the segregation case, prove properties of the limiting problem.

In the spin orbit coupling segregation case, get the Γ convergence to understand the disk+annuli and stripes configurations.

In the rotation segregation case, get the Γ convergence to understand the stripes

In the rotating coexistence case, get more precise energy estimates for the vortex/spike problem.

In the rotating coexistence case, analyze the lattice using Theta functions