Estimating occupation time of continuous semimartingales

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OCCUPATION TIME

- d-dimensional process $X = (X_t)_{0 \le t \le 1}$, function $f : \mathbb{R}^d \to \mathbb{R}$
- **occupation time**:

$$\Gamma_t(f) := \int_0^t f(X_r)\, dr, \;\; 0 \le t \le 1$$

- *Example.* $f = \mathbf{1}_A$, $A \in \mathscr{B}_{\mathbb{R}^d}$,

$$\Gamma_t(f) = \int_0^t \mathbf{1}_A(X_r)\, dr$$

$\Rightarrow$ describes how much time X spends in A

ESTIMATION PROBLEM

- given: discrete sample $\left(X_{\frac{k}{n}}\right)_{k=0,\dots,n}$
- estimate $\Gamma_t(f) = \int_0^t f(X_r)dr$ by *Riemann-estimator*

$$\hat{\Gamma}_{n,t}(f) = \frac{1}{n}\sum_{k=1}^{\lfloor nt \rfloor} f\left(X_{\frac{k-1}{n}}\right)$$

- toy example: paths of X and function f are C^1 such that

$$\left|\Gamma_t(f) - \hat{\Gamma}_{n,t}(f)\right| = \left|\sum_{k=1}^{\lfloor nt \rfloor}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(X_r) - f\left(X_{\frac{k-1}{n}}\right)\right)dr\right|$$

$$= \left|\sum_{k=1}^{\lfloor nt \rfloor}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\int_{\frac{k-1}{n}}^{r} f'(X_h)X_h' dh\right)dr\right|$$

$$\leq \frac{C}{n}$$

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- toy example: paths of X and function f are C^1 such that

$$\begin{aligned}\left|\Gamma_t(f) - \hat{\Gamma}_{n,t}(f)\right| &= \left|\sum_{k=1}^{\lfloor nt \rfloor}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(X_r) - f\left(X_{\frac{k-1}{n}}\right)\right)dr\right| \\ &= \left|\sum_{k=1}^{\lfloor nt \rfloor}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\int_{\frac{k-1}{n}}^{r} f'(X_h)X_h' dh\right)dr\right| \\ &\leq \frac{C}{n}\end{aligned}$$

RESULTS FROM THE LITERATURE

- **Ngo and Ogawa (2011)**: $d = 1$, X is "nice" diffusion, $f = \mathbf{1}_{[a,b]}$
  - $\hat{\Gamma}_{n,t}\left(\mathbf{1}_{[a,b]}\right) - \Gamma_t\left(\mathbf{1}_{[a,b]}\right) = O_{\mathbb{P}}\left(n^{-3/4}\right)$
  - rate $n^{3/4}$ optimal

- **Kohatsu-Higa et al. (2014)**: $d = 1$, X is "nice" diffusion
  - $\hat{\Gamma}_{n,t}(f) - \Gamma_t(f) = \begin{cases} O_{\mathbb{P}}\left(n^{-\frac{1+\alpha}{2}}\right), & \text{if } f \ \alpha\text{-Hölder}, \alpha \in (0,1) \\ O_{\mathbb{P}}\left(n^{-3/4}\right), & \text{if } f \text{ in some approx. class } \mathscr{A} \end{cases}$

- **Ganychenko and Kulik (2014)**: general d, X is Markov process, conditions on transition density
  - e.g. if f bounded, then $\hat{\Gamma}_{n,t}(f) - \Gamma_t(f) = O_{\mathbb{P}}\left((\log n)^{1/2} n^{-\frac{1}{2}}\right)$
  - e.g. if X is diffusion with smooth coefficients, f α-Hölder, then $\hat{\Gamma}_{n,t}(f) - \Gamma_t(f) = O_{\mathbb{P}}\left((\log n)^{1/2} n^{-\left(\frac{1}{2}+\frac{\alpha}{4}\right)}\right)$

OVERVIEW

1. "direct approach":
   - $d \geq 1$, X is Brownian motion
   - unifying rates of convergence

2. using Itô formula:
   - $d \geq 1$, X is continuous semimartingale
   - (stable) CLT

3. generator approach
   - $d \geq 1$, X is stationary reversible Markov process
   - wait for Jakub's talk...

DIRECT APPROACH

- $X = W$ is Brownian motion, $f \in C_c^\infty(\mathbb{R}^d)$, $t = 1$; then

$$\|\hat{\Gamma}_{1,n}(f) - \Gamma_1(f)\|^2_{L^2(\mathbb{P})} = \mathbb{E}\left[\left|\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(f(W_r) - f\left(W_{\frac{k-1}{n}}\right)\right) dr\right|^2\right]$$

$$= \sum_{j,k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \mathbb{E}\left[\left(f(W_r) - f\left(W_{\frac{k-1}{n}}\right)\right)\left(f(W_h) - f\left(W_{\frac{j-1}{n}}\right)\right)\right] drdh$$

- by inverse Fourier transform:

$$f(W_r) - f\left(W_{\frac{k-1}{n}}\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \mathscr{F}f(u) \left(e^{-i\langle u, W_r\rangle} - e^{-i\left\langle u, W_{\frac{k-1}{n}}\right\rangle}\right) du$$

DIRECT APPROACH

- by plug-in for $0 < s \leq 1$, $\beta = \frac{1+s}{2} \wedge 1$,

$$\begin{aligned}
&\|\hat{\Gamma}_{1,n}(f) - \Gamma_1(f)\|^2_{L^2(\mathbb{P})} \\
&\lesssim \frac{1}{n^{2\beta+2}} \sum_{j,k=1}^{n} \int_{\mathbb{R}^{2d}} |\mathscr{F} f(u) \mathscr{F} f(v)| \, |u|^{2\beta} \, |v|^{2\beta} \, e^{-\frac{|u|^2}{2} \frac{k-j+1}{n}} e^{-\frac{|u+v|^2}{2} \frac{j-1}{n}} dudv \\
&\lesssim \frac{\|f\|^2_{H^s}}{n^{2\beta}} \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{\left(\frac{k}{n}\right)^{2\beta - s}} \right) \left(\frac{1}{n} \sum_{j=1}^{n} \frac{1}{\left(\frac{j}{n}\right)^{d/2}} \right)
\end{aligned}$$

- *fractional Sobolev spaces*: $s > 0$,

$$\begin{aligned}
H^s(\mathbb{R}^d) &= \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{H^s} < \infty \right\}, \\
\|f\|_{H^s} &= \|(1 + |\cdot|)^s \mathscr{F} f(\cdot)\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

DIRECT APPROACH

- $\Gamma_{t_1,t_2}(f) = \int_{t_1}^{t_2} f(X_r)dr$, $\hat{\Gamma}_{t_1,t_2,n}(f) = \frac{1}{n}\sum_{k=\lfloor nt_1\rfloor+1}^{\lfloor nt_2\rfloor} f(X_{\frac{k-1}{n}})$

Theorem

For $0 \leq t_1 \leq t_2 \leq 1$, $f \in H^s(\mathbb{R}^d)$, $X = W$*:*

1. $0 < t_1$, $0 < s \leq 1$*:*

$$\|\hat{\Gamma}_{t_1,t_2,n}(f) - \Gamma_{t_1,t_2}(f)\|_{L^2(\mathbb{P})} \lesssim \frac{\|f\|_{H^s} (\log n)^{1/2}}{n^{\frac{1+s}{2}}}.$$

2. $0 < t_1$, $s > 1$*:*

$$\|\hat{\Gamma}_{t_1,t_2,n}(f) - \Gamma_{t_1,t_2}(f)\|_{L^2(\mathbb{P})} \lesssim \frac{\|f\|_{H^s}}{n}.$$

3. $d = 1$ *or* $X_t = X_0 + W_t$, $X_0 \overset{d}{\sim} N(0, aI)$, $a > 0$*: 1 and 2 also hold for* $t_1 = 0$.

CLT: A FUNDAMENTAL DECOMPOSITION

- $X_t = \int_0^t b_r dr + \int_0^t \sigma_r dW_r$, for simplicity $d = 1$
- we have

$$\Gamma_t(f) - \hat{\Gamma}_{n,t}(f) = \sum_{k=1}^{\lfloor nt \rfloor} Z_k = \underbrace{\sum_{k=1}^{\lfloor nt \rfloor} \left(Z_k - \mathbb{E}\left[Z_k \middle| \mathscr{F}_{\frac{k-1}{n}} \right] \right)}_{discrete\ martingale} + \underbrace{\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\left[Z_k \middle| \mathscr{F}_{\frac{k-1}{n}} \right]}_{drift},$$

$$Z_k = \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(f(X_r) - f\left(X_{\frac{k-1}{n}} \right) \right) dr$$

- if $f \in C^2$, then by Itô's formula

$$Z_k = \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(\int_{\frac{k-1}{n}}^{r} f'(X_h)\, dX_h + \frac{1}{2} \int_{\frac{k-1}{n}}^{r} f''(X_h)\, d\langle X \rangle_h \right) dr,$$

$$\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\left[Z_k \middle| \mathscr{F}_{\frac{k-1}{n}} \right] \approx \frac{1}{4n^2} \sum_{k=1}^{\lfloor nt \rfloor} f''\left(X_{\frac{k-1}{n}} \right) \sigma_{\frac{k-1}{n}} \approx \frac{1}{4n} \int_0^t f''(X_r)\, d\langle X \rangle_r,$$

$$\sum_{k=1}^{\lfloor nt \rfloor} \left(\mathbb{E}\left[Z_k^2 \middle| \mathscr{F}_{\frac{k-1}{n}} \right] - \left(\mathbb{E}\left[Z_k \middle| \mathscr{F}_{\frac{k-1}{n}} \right] \right)^2 \right) \approx \frac{1}{3n^3} \sum_{k=1}^{\lfloor nt \rfloor} \left(f'\left(X_{\frac{k-1}{n}} \right) \right)^2 \approx \frac{1}{3n^2} \int_0^t \left(f'(X_r) \right)^2 d\langle X \rangle_r.$$

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CLT FOR C^2-FUNCTIONS

Theorem

If $f \in C^2$, then we have the stable convergence (on $D([0,1])$)

$$n\left(\hat{\Gamma}_{n,t}(f) - \Gamma_t(f)\right) \xrightarrow{st} -\frac{1}{2}\int_0^t f'(X_r)\,dX_r - \frac{1}{4}\int_0^t f''(X_r)\,d\langle X\rangle_r + \int_0^t \frac{1}{\sqrt{12}}\left(f'(X_r)\sigma_r\right)d\tilde{W}_r = -\frac{1}{2}\left(f(X_t) - f(X_0)\right) + \int_0^t \frac{1}{\sqrt{12}}\left(f'(X_r)\sigma_r\right)d\tilde{W}_r,$$

where $\tilde{W} \perp (\Omega, \mathscr{F}, \mathbb{P})$.

BEYOND C^2-FUNCTIONS

- for $f \in C_c^\infty$ by inverse Fourier transform for $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_{t_1}^{t_2} f''(X_h)\, d\langle X\rangle_h = -\frac{1}{2\pi}\int_{\mathbb{R}} \mathscr{F}f(u)\, u^2 \left(\int_{t_1}^{t_2} e^{-iuX_h}\, d\langle X\rangle_h\right) du,$$

$$\xrightarrow{\text{Hölder}} |\cdot| \lesssim \|(1+|\cdot|)^s \mathscr{F}f(\cdot)\|_{L^p}\ \ \|(1+|\cdot|)^{2-s}\int_{t_1}^{t_2} e^{-i(\cdot)X_h}\, d\langle X\rangle_h\|_{L^q}$$

- Fourier-Lebesgue space: $p \geq 1, s \in \mathbb{R}$

$$\begin{aligned} FL^{s,p}(\mathbb{R}^d) &= \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{FL^{s,p}} < \infty \right\}, \\ \|f\|_{FL^{s,p}} &= \|(1+|\cdot|)^s \mathscr{F}f(\cdot)\|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

- if $A \mapsto \Phi^X_{t_1,t_2}(A) = \int_{t_1}^{t_2} \mathbf{1}_A(X_r)\, d\langle X\rangle_r$ denotes the occupation measure of X, then

$$\int_{t_1}^{t_2} e^{-iuX_h}\, d\langle X\rangle_h = \int_{\mathbb{R}} e^{iuy}\, \Phi^X_{t_1,t_2}(dy) = \left(\mathscr{F}\Phi^X_{t_1,t_2}\right)(u)$$

is precisely its Fourier transform

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GENERALIZED ITÔ FORMULA

Lemma

Grant "some" smoothness assumptions on σ. For any $2 \leq q \leq \infty$, $\gamma < 1 - d/q$, we have

$$\|\Phi^X_{t_1,t_2}\|_{FL^{\gamma,q}} = \|(1+|\cdot|)^\gamma \int_{t_1}^{t_2} e^{-i\langle \cdot, X_h\rangle} d\langle X\rangle_h\|_{L^q(\mathbb{R}^d)} < \infty, \quad a.s.$$

- "Some" smoothness assumptions:
  - $\sigma_t \sigma_t^\top > 0$ for $0 < t \leq 1$
  - $\mathbb{E}\left[\sup_{0\leq\delta\leq s} \|\sigma_{\delta+r} - \sigma_r\|^q\right] \leq C_q s^{q\alpha}$ for $q \geq 2$, $\alpha \geq 1/2$ and for all $0 \leq r+s \leq T$, $r \geq 0$
- important approximation (inspired by Fournier et al. (2008)) : $u \in \mathbb{R}^d$, $h > 0$, $\varepsilon = \varepsilon(u) > 0$,

$$\mathbb{E}\left[e^{-i\langle u, X_h\rangle}\right] \approx \mathbb{E}\left[e^{-i\langle u, X_{h-\varepsilon} + b_{h-\varepsilon}\varepsilon + \sigma_{h-\varepsilon}(W_h - W_{h-\varepsilon})\rangle}\right]$$

GENERALIZED ITÔ FORMULA

Theorem

Grant "some" smoothness assumptions on σ. Let $0 \leq t_1 \leq t_2 \leq 1$, $p \geq 1, q \geq 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in FL^{s,p}(\mathbb{R}^d)$ with $s > 1 + d/q$, then the function f satisfies the Itô formula

$$f(X_{t_2}) = f(X_{t_1}) + \int_{t_1}^{t_2} \langle \nabla f(X_r), dX_r \rangle + \frac{1}{2} \mathscr{L}^X_{t_1,t_2}(f),$$

where $\mathscr{L}^X_{t_1,t_2}(f) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathscr{F} f(u)\, u_i u_j \left(\int_{t_1}^{t_2} e^{-i\langle u, X_r \rangle} d\langle X_i, X_j \rangle_r \right) du.$

CLT FOR SOBOLEV-FUNCTIONS

Theorem

Grant "some" smoothness assumptions on σ. Let $p \geq 1, q \geq 2$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in FL^{s,p}(\mathbb{R}^d)$ with $s > 1 + d/q$, then we have the stable convergence (on $D([0,1])$)

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where $\tilde{W} \perp (\Omega, \mathscr{F}, \mathbb{P})$.

SUMMARY

1. "direct approach":
   - $d \geq 1$, X is Brownian motion, $f \in H^s$
   - unifying rates of convergence: e.g. $0 < s \leq 1$,
     $\|\hat{\Gamma}_{t_1,t_2,n}(f) - \Gamma_{t_1,t_2}(f)\|_{L^2(\mathbb{P})} \lesssim \frac{\|f\|_{H^s}(\log n)^{1/2}}{n^{\frac{1+s}{2}}}$

2.a generalized Itô's formula: $f \in FL^{s,p}(\mathbb{R}^d)$ with $s > 1 + d/q$
   - $f(X_{t_2}) = f(X_{t_1}) + \int_{t_1}^{t_2} \langle \nabla f(X_r), dX_r \rangle + \frac{1}{2} \mathscr{L}^X_{t_1,t_2}(f)$.

2.b using (generalized) Itô formula:
   - $d \geq 1$, X is cont. semimart., $f \in FL^{s,p}(\mathbb{R}^d)$ with $s > 1 + d/q$
   - (stable) CLT

3. generator approach:
   - $d \geq 1$, X is stationary reversible Markov process
   - Jakub's talk is coming up!

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