Introduction to random fields and scale invariance: Lecture I

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Outlines

Lecture 1 :

1. Introduction to random fields
   1. Definitions and law
   2. Gaussian processes
   3. Gaussian fields from processes
2. Stationarity and Invariances
   1. Stationarity and Isotropy
   2. Self-similarity or scale invariance
   3. Stationary increments
   4. Operator scaling Property

Introduction to random fields

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and for $d \geq 1$, $T \subset \mathbb{R}^d$ is a set of indices

Definition

A (real) stochastic process indexed by T is just a collection of real random variables $X_t : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable, $\forall t \in T$.

Exples :

- $d = 1$, $X_t(\omega)$ =heart frequency at time $t \in T \subset \mathbb{R}$, with noise measurement or for an individual ω. In practice data are only available on a discrete finite subset S of T
- $d = 2$, $T = [0, 1]^2$, $X_t(\omega)$ = grey level of a picture at point $t \in T$. In practice data are only available on pixels $S = \{0, 1/n, \dots, 1\}^2 \subset T$ for an image of size $(n + 1) \times (n + 1)$.

Introduction to random fields

Definition

The distribution of $(X_t)_{t\in T}$ *is given by all its finite dimensional distribution (fdd) ie the distribution of all real random vectors*

$$(X_{t_1} \ldots, X_{t_k}) \text{ for } k \geq 1, t_1, \ldots, t_k \in T.$$

Joint distributions are often difficult to compute !

Definition

$(X_t)_{t\in T}$ *is a second order of process if* $\mathbb{E}(X_t^2) < +\infty$ *for all* $t \in T$.

- *Mean function* $m_X : t \in T \to \mathbb{E}(X_t) \in \mathbb{R}$
- *Covariance function* $K_X : (t,s) \in T \times T \to Cov(X_t, X_s) \in \mathbb{R}$.

Introduction to random fields

When $m_X = 0$, the process X is centered. Otherwize $Y = X - m_X$ is centered and $K_Y = K_X$.

Proposition

A function $K : T \times T \to \mathbb{R}$ is a covariance function iff

1. *K is symmetric*
2. *K is positive definite : $\forall k \geq 1, t_1, \ldots, t_k \in T, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$,*

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j K(t_i, t_j) \geq 0.$$

Gaussian Processes

Definition

$(X_t)_{t\in T}$ is a Gaussian process if $\forall k \geq 1, t_1, \ldots, t_k \in T$

$$(X_{t_1}, \ldots, X_{t_k}) \text{ is a Gaussian vector of } \mathbb{R}^k,$$

EQ $\forall \lambda_1, \ldots, \lambda_k \in \mathbb{R}$, *the real random variable* $\sum_{i=1}^{k} \lambda_i X_{t_i}$ *is a Gaussian variable.*

Proposition

When $(X_t)_{t\in T}$ is a Gaussian process, $(X_t)_{t\in T}$ is a second order process and its law is determined by its mean function $m_X : t \mapsto \mathbb{E}(X_t)$ and its covariance function $K_X : (t,s) \mapsto Cov(X_t, X_s)$.

Theorem (Komogorov)

Let $m : T \to \mathbb{R}$ and $K : T \times T \to \mathbb{R}$, symmetric and positive definite, then there exists a Gaussian process with mean m and covariance K.

Brownian motion on $\mathbb{R}^+$

$T = \mathbb{R}^+$ and $(X_k)_k$ iid $\mathbb{E}(X_k) = 0$ and $\text{Var}(X_k) = 1$

$$\forall t \in T, S_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k.$$

By CLT $S_n(t) \xrightarrow[n\to+\infty]{d} \mathcal{N}(0,t)$. Moreover, if $t_0 = 0 < t_1 < \ldots < t_k$,

$$(S_n(t_1), S_n(t_2) - S_n(t_1), \ldots, S_n(t_k) - S_n(t_{k-1})) \xrightarrow[n\to+\infty]{d} Z = (Z_1, \ldots, Z_k),$$

with $Z \sim \mathcal{N}(0, K_Z)$ for $K_Z = \text{diag}(t_1, t_2 - t_1, \ldots, t_k - t_{k-1})$. Hence

$$\begin{aligned} & (S_n(t_1), S_n(t_2), \ldots, S_n(t_k)) \\ = \; & P(S_n(t_1), S_n(t_2) - S_n(t_1), \ldots, S_n(t_k) - S_n(t_{k-1})) \\ \xrightarrow[n\to+\infty]{d} \; & PZ, \end{aligned}$$

with $PZ \sim \mathcal{N}(0, PK_ZP^*)$ and $PK_ZP^* = (\min(t_i, t_j))_{1\le i,j\le k}$.

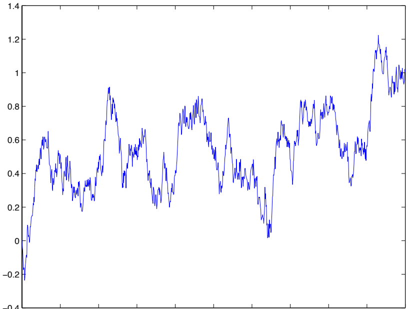
Brownian motion on $\mathbb{R}$

Note that $K(t,s) = \min(t,s) = \frac{1}{2}(t + s - |t - s|)$ is a cov. func. on $\mathbb{R}^+ \times \mathbb{R}^+$. Let $B_t = X_t^{(1)}$ for $t \geq 0$, $B_t = X_{-t}^{(2)}$ for $t < 0$ with $X^{(1)}$ and $X^{(2)}$ 2 iid K.

Definition

A (standard) Brownian motion on $\mathbb{R}$ is a centered Gaussian process $(B_t)_{t\in\mathbb{R}}$ with covariance function given by

$$Cov(B_t, B_s) = \frac{1}{2}(|t| + |s| - |t - s|), \ \forall t, s \in \mathbb{R}.$$



Gaussian fields from processes

Proposition

Let $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous covariance function. For all μ positive finite measure on S^{d-1}

$$(x, y) \mapsto \int_{S^{d-1}} K(x \cdot \theta, y \cdot \theta) d\mu(\theta),$$

is a covariance function on $\mathbb{R}^d \times \mathbb{R}^d$.

Exple : Note that $\int_{S^{d-1}} |x \cdot \theta| d\theta = c_d \|x\|$, with $c_d = \int_{S^{d-1}} |e \cdot \theta| d\theta$ for $e = (1, 0, \ldots, 0)$. Then,

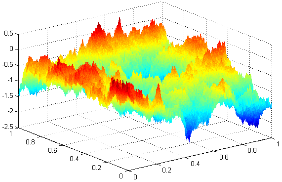
$$\int_{S^{d-1}} K_B(x \cdot \theta, y \cdot \theta) d\theta = \frac{c_d}{2} \left(\|x\| + \|y\| - \|x - y\| \right).$$

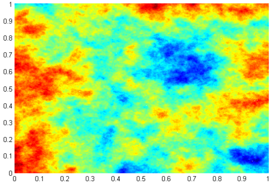
Lévy Chentsov random field

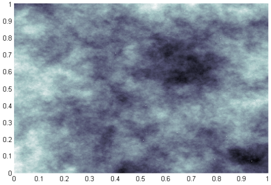
Definition

A (standard) Levy Chentsov field on $\mathbb{R}^d$ *is a centered Gaussian field* $(X_x)_{x\in\mathbb{R}^d}$ *with covariance function given by*

$$Cov(X_x, X_y) = \frac{1}{2}\left(\|x\| + \|y\| - \|x-y\|\right), \ \forall x, y \in \mathbb{R}^d.$$







Gaussian fields from processes

Proposition

Let $K_1, K_2, \dots, K_d$ covariance functions on $\mathbb{R} \times \mathbb{R}$, then

$$(x, y) \mapsto \prod_{i=1}^{d} K_i(x_i, y_i),$$

is a covariance function on $\mathbb{R}^d \times \mathbb{R}^d$.

Exple : Brownian sheet $(x, y) \mapsto \prod_{i=1}^{d} \frac{1}{2}(|x_i| + |y_i| - |x_i - y_i|)$

Stationarity

Definition

$X = (X_x)_{x \in \mathbb{R}^d}$ *(strongly) stationary if,* $\forall x_0 \in \mathbb{R}^d$, $(X_{x+x_0})_{x \in \mathbb{R}^d}$ *has the same law than* X.

Proposition

If $X = (X_x)_{x \in \mathbb{R}^d}$ *stationary and second order,* $\forall x_0 \in \mathbb{R}^d$,

- $m_X(x) = m_X$
- $K_X(x, y) = c_X(x - y)$ *with* $c_X : \mathbb{R}^d \to \mathbb{R}$ *s.t.*
  1. $c_X(0) \geq 0$
  2. $|c_X(x)| \leq c_X(0)\ \forall x \in \mathbb{R}^d$
  3. c_X *is of positive type ie*
     $\forall k \geq 1, x_1, \ldots, x_k \in \mathbb{R}^d, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$,
     $$\sum_{i,j=1}^{k} \lambda_i \lambda_j c_X(x_i - x_j) \geq 0.$$

Stationarity

Theorem (Bochner 1932)

An even continuous function $c : \mathbb{R}^d \to \mathbb{R}$ is of positive type if and only if $c(0) > 0$ and there exists a symmetric probability measure ν on $\mathbb{R}^d$ such that

$$c(x) = c(0) \int_{\mathbb{R}^d} e^{it\cdot x} d\nu(x).$$

In other words there exists a symmetric random vector Z on $\mathbb{R}^d$ such that

$$c(x) = c(0)\mathbb{E}(e^{ix\cdot Z}).$$

Rk : When c_X is the covariance of the stationary field X, ν_X is called the spectral measure of X.

Ornstein Uhlenbeck process

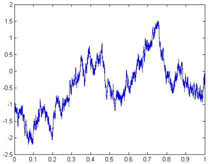
Let B a Brownian motion on $\mathbb{R}^+$, $\theta > 0$ and define

$$X_t = e^{-\theta t} B_{e^{2\theta t}},$$

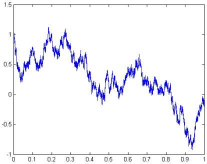
then X is a centered stationary Gaussian process on $\mathbb{R}$ with covariance

$$\mathrm{Cov}(X_t, X_s) = e^{-\theta|t-s|}, \ \forall t, s \in \mathbb{R},$$

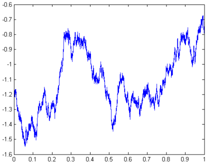
and $\nu_X(dt) = \frac{\theta^2}{\pi(\theta^2+t^2)} dt$.



$\theta = 5$



$\theta = 1$



$\theta = 1/5$

Isotropy

Definition

$X = (X_x)_{x\in\mathbb{R}^d}$ isotropic if, $\forall R$ rotation, $(X_{Rx})_{x\in\mathbb{R}^d}$ has the same law than X.

Exple : the Lévy Chentsov field is isotropic since

$$\begin{aligned}\mathrm{Cov}(X_{Rx}, X_{Ry}) &= \frac{1}{2}\left(\|Rx\| + \|Ry\| - \|Rx - Ry\|\right)\\ &= \mathrm{Cov}(X_x, X_y)\end{aligned}$$

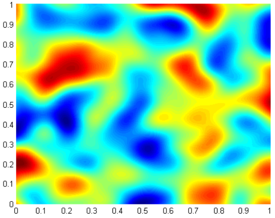
Exple : If $g(t) = e^{-t^2/2}$ then $k(t,s) = g(t-s)$ covariance and

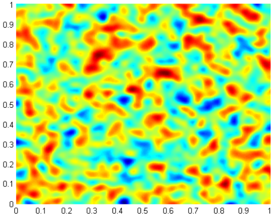
$$K(x,y) = e^{-\|x-y\|^2/2} = \prod_{i=1}^{d} k(x_i, y_i),$$

allows to define a stationary isotropic Gaussian field.

Isotropy

Gaussian covariance





$$K(x,y) = e^{-100\|x-y\|^2} \qquad K(x,y) = e^{-1000\|x-y\|^2}$$

[Powell, LNS, 2014]

Self-similarity

Definition

$X = (X_x)_{x\in\mathbb{R}^d}$ self-similar of order $H > 0$ if, $\forall c > 0$, $(X_{cx})_{x\in\mathbb{R}^d}$ has the same law than $c^H X$.

Exple : the Lévy Chentsov field is self-similar of order $H = 1/2$ since

$$\begin{aligned}\mathrm{Cov}(X_{cx}, X_{cy}) &= \frac{1}{2}\left(\|cx\| + \|cy\| - \|cx - cy\|\right)\\ &= c\mathrm{Cov}(X_x, X_y) = \mathrm{Cov}(c^{1/2}X_x, c^{1/2}X_y)\end{aligned}$$

Corollary

There does not exist a (non-trivial) stationary self-similar field.

Stationary increments

Definition

$X = (X_x)_{x\in\mathbb{R}^d}$ *has (strongly) stationary increments if,* $\forall x_0 \in \mathbb{R}^d$, $(X_{x+x_0} - X_{x_0})_{x\in\mathbb{R}^d}$ *has the same law than* $(X_x - X_0)_{x\in\mathbb{R}^d}$.

Proposition

If $X = (X_x)_{x\in\mathbb{R}^d}$ *second order centered with s.i. and* $X_0 = 0$ *a.s.,*

- $K_X(x,y) = \frac{1}{2}\left(v_X(x) + v_X(y) - v_X(x-y)\right),\ \forall x,y \in \mathbb{R}^d$
- $v_X(x) = Var(X_{x+x_0} - X_{x_0}) = Var(X_x)$ *called variogram s.t.*
  1. $v_X(0) = 0$
  2. $v_X(x) \geq 0$ *and* $v_X(-x) = v_X(x)$
  3. v_X *is conditionally of negative type ie* $\forall k \geq 1, x_1,\ldots,x_k \in \mathbb{R}^d, \lambda_1,\ldots,\lambda_k \in \mathbb{R}$,

$$\sum_{i=1}^{k}\lambda_i = 0 \Rightarrow \sum_{i,j=1}^{k}\lambda_i\lambda_j v_X(x_i - x_j) \leq 0.$$

Stationary increments

Theorem (Schoenberg)

Let $v : \mathbb{R}^d \to \mathbb{R}$ be an even continuous function. EQU

i) v *is conditionally of negative type*

ii) $K : (x, y) \mapsto \frac{1}{2}(v(x) + v(y) - v(x - y))$ *is a covariance function*

iii) $\forall \lambda > 0$, $e^{-\lambda v}$ *is of positive type*

Corollary (Istas, 2006)

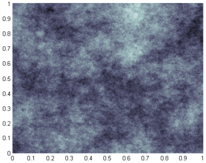
If v is a variogram then v^H is a variogram $\forall H \in (0, 1]$.

Fractional Brownian fields

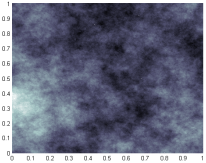
Definition

A (standard) Fractional Brownian field on $\mathbb{R}^d$*, with Hurst parameter* $H \in (0,1)$*, is a centered Gaussian field* $(B_H)_{x\in\mathbb{R}^d}$ *with covariance function given by*

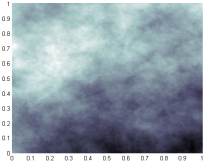
$$Cov(B_H(x), B_H(y)) = \frac{1}{2}\left(\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\right), \ \forall x, y \in \mathbb{R}^d.$$



$H = 0.2$



$H = 0.4$



$H = 0.6$

Fractional Brownian fields

Main Properties :

- stationary increments : $\forall x_0 \in \mathbb{R}^d,\ B_H(x_0 + \cdot) - B_H(x_0) \overset{fdd}{=} B_H(\cdot)$
- H self-similarity : $\forall c > 0,\ B_H(c\cdot) \overset{fdd}{=} c^H B_H(\cdot)$
- Isotropy : $\forall R$ rotation $B_H(R\cdot) \overset{fdd}{=} B_H(\cdot)$

➥Uniqueness up to a constant

Remarks :

- for $d = 1$ called fractional Brownian motion [Kolmogorov, 1940], [Mandelbrot and Van Ness, 1968]
- sssi implies that $H \leq 1$
- (isotropic) sssi for $H = 1$ corresponds to $(Z \cdot x)_{x \in \mathbb{R}^d}$ with Z (isotropic) Gaussian vector on $\mathbb{R}^d$.

Anisotropic generalizations

Let $H \in (0,1)$ and $v_H : t \in \mathbb{R} \mapsto |t|^{2H}$, conditionally of negative type. If μ is a finite positive measure on S^{d-1},

$$v_{H,\mu}(x) = \int_{S^{d-1}} v_H(x\cdot\theta)\mu(d\theta) = \int_{S^{d-1}} |x\cdot\theta|^{2H}\mu(d\theta) = c_{H,\mu}\left(\frac{x}{\|x\|}\right)\|x\|^{2H},$$

is conditionally of negative type function on $\mathbb{R}^d$.

Let $X_{H,\mu} = (X_{H,\mu}(x))_{x\in\mathbb{R}^d}$ be a centered Gaussian random field with s.i. and variogram $v_{H,\mu}$, it is still H s.s. but may not be isotropic

➥ $c_{H,\mu}$ is called topothesy function

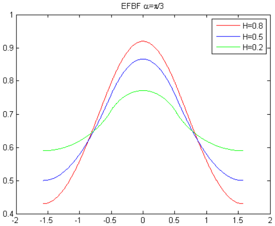
Exple : Let $d = 2$ and for $\alpha \in (0, \pi/2]$, $\mu(d\theta) = \mathbf{1}_{(-\alpha,\alpha)}(\theta)d\theta$

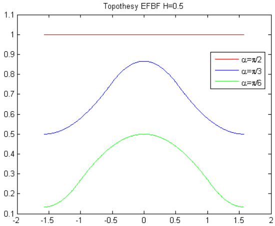
Elementary anisotropic fractional Brownian fields

Then $c_{H,\alpha}$ is a π periodic function defined on $(-\pi/2, \pi/2]$ by

$$c_{H,\alpha}(\theta) = 2^{2H} \begin{cases} \beta_H\left(\frac{1-\sin(\alpha-\theta)}{2}\right) + \beta_H\left(\frac{1+\sin(\alpha+\theta)}{2}\right) & \text{if} \quad -\alpha \leq \theta + \frac{\pi}{2} \leq \alpha \\ \beta_H\left(\frac{1+\sin(\alpha-\theta)}{2}\right) + \beta_H\left(\frac{1-\sin(\alpha+\theta)}{2}\right) & \text{if} \quad -\alpha \leq \theta - \frac{\pi}{2} \leq \alpha \\ \left|\beta_H\left(\frac{1-\sin(\alpha-\theta)}{2}\right) - \beta_H\left(\frac{1+\sin(\alpha+\theta)}{2}\right)\right| & \text{otherwise} \end{cases}$$

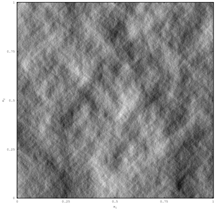
with $\beta_H(t) = \int_0^t u^{H-1/2}(1-u)^{H-1/2}du$ is a Beta incomplete function.



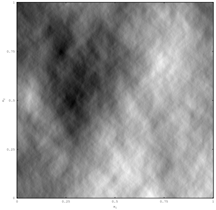


[HB, Moisan, Richard, 2015]

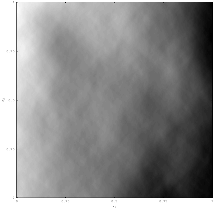
Elementary anisotropic fractional Brownian fields



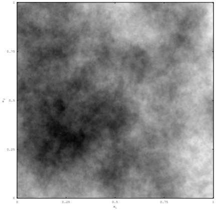
$H = 0.2,\ \alpha = \pi/3$



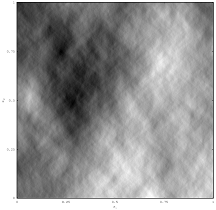
$H = 0.5,\ \alpha = \pi/3$



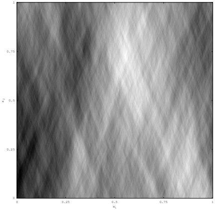
$H = 0.8,\ \alpha = \pi/3$



$\alpha = \pi/2,\ H = 0.5$



$\alpha = \pi/3,\ H = 0.5$



$\alpha = \pi/6,\ H = 0.5$

Operator scaling random fields

Let E be a $d \times d$ diagonalizable matrix with eigenvalues $\alpha_1^{-1}, \ldots, \alpha_d^{-1} \in [1, +\infty)$ and $\theta_1, \ldots, \theta_d$ be such that $E^t\theta_i = \alpha_i^{-1}\theta_i$.
For $H \in (0, 1]$, we define the variogram

$$v_{H,E}(x) = \tau_E(x)^{2H} = \left(\sum_{i=1}^{d} |\langle x, \theta_i\rangle|^{2\alpha_i}\right)^H = \left(\sum_{i=1}^{d} v_{\alpha_i}(\langle x, \theta_i\rangle)\right)^H .$$

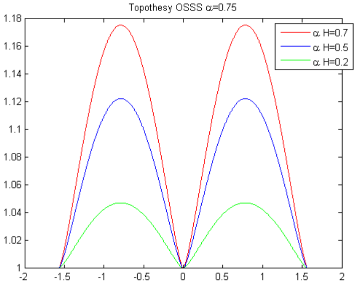
Let $X_{H,E} = (X_{H,E}(x))_{x\in\mathbb{R}^d}$ be a centered Gaussian random field with s.i. and variogram $v_{H,E}$. Then, it is (E, H) operator scaling :

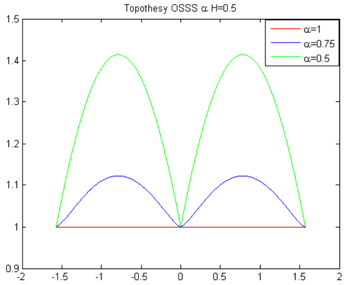
$$\forall c > 0,\ X_{H,E}(c^E\cdot) \stackrel{fdd}{=} c^H X_{H,E}(\cdot).$$

[HB, Meerschaert, Scheffler, 2007] & [HB, Lacaux, in preparation]

SS Operator scaling random fields

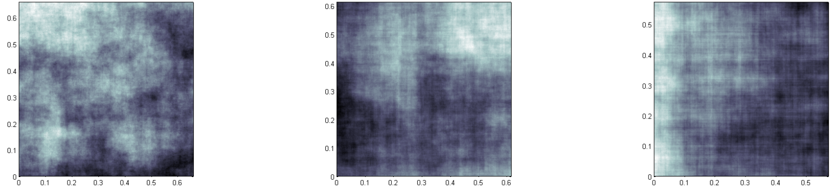
When $\alpha_1 = \ldots = \alpha_d = \alpha \in (0,1]$, $X_{H,E}$ is αH self-similar.



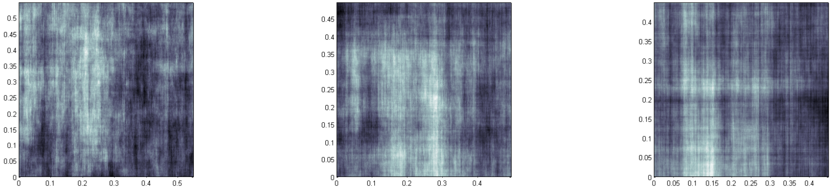


SS Operator scaling random fields

Self-similar of order $H\alpha_1 = H\alpha_2 = 0.5$



Operator scaling with $H\alpha_1 = 0.3$ and $H\alpha_2 = 0.4$



$H = 0.6$ $H = 0.7$ $H = 0.8$

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