Introduction to random fields and scale invariance: Lecture III

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Outlines

Lecture 3 :

1. Simulation
   1. Simulation of FBM
   2. Turning band method for Anisotropic SS fields
   3. Stein method for Operator Scaling fields
2. Estimation
   1. 1D estimation based on variograms
   2. Application to random fields by line processes

Fast and exact synthesis of 1d fBm

Let $H \in (0,1)$ and B_H a fBm :

- by self-similarity, for all $c > 0$,

$$(B_H(ck))_{0 \le k \le n} \stackrel{d}{=} c^H (B_H(k))_{0 \le k \le n}.$$

- since $B_H(0) = 0$ a.s., $B_H(k) = \sum_{j=0}^{k-1} (B_H(j+1) - B_H(j))$ for $k \ge 1$

For $j \in \mathbb{Z}$, the fractional gaussian noise is defined as

$$Y_j = B_H(j+1) - B_H(j)$$

so that $(Y_j)_{j \in \mathbb{Z}}$ is a centered stationary Gaussian sequence with

$$c_k = \mathrm{Cov}(Y_{k+j}, Y_j) = \frac{1}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right), \forall k \in \mathbb{Z}.$$

Circulant embedding matrix

[Dietrich & Newsam, 97] Let $Y = (Y_0, \ldots, Y_n) \sim \mathcal{N}(0, K_Y)$ with

$$K_Y = \begin{pmatrix} c_0 & c_1 & \ldots & c_n \\ & \ddots & & \vdots \\ & & \ddots & c_1 \\ & & & c_0 \end{pmatrix}, \text{ since } \mathrm{Cov}(Y_{k+j}, Y_j) = c_k$$

Embed in the symmetric circulant matrix $S = circ(s)$ of size $2n$ with

$$s = (c_0\; c_1\; \ldots\; c_n\; c_{n-1}\; \ldots\; c_1) = (s_0\; s_1\; \ldots\; s_n\; s_{n+1}\; \ldots\; s_{2n-1})$$

ie

$$S = \begin{pmatrix} s_0 & s_{2n-1} & \ldots & s_2 & s_1 \\ s_1 & s_0 & s_{2n-1} & & s_2 \\ \vdots & s_1 & s_0 & \ddots & \vdots \\ s_{2n-2} & & \ddots & \ddots & s_{2n-1} \\ s_{2n-1} & s_{2n-2} & \ldots & s_1 & s_0 \end{pmatrix} = \begin{pmatrix} K_Y & S_1 \\ S_1^t & S_2 \end{pmatrix}$$

Circulant embedding matrix

Then $S = \frac{1}{2n} F_{2n}^* \text{diag}(F_{2n} s) F_{2n}$ with F_{2n} the matrix of discrete Fourier transform.

Theorem [Perrin et al, 2002, Craigmile, 2003] : S is a covariance matrix ($\Leftrightarrow F_{2n} s \geq 0$).

Let $R_{2n} = \frac{1}{\sqrt{2n}} F_{2n}^* \text{diag}(F_{2n} s)^{1/2} \in \mathcal{M}_{2n}(\mathbb{C})$ then, for $\varepsilon^{(1)}, \varepsilon^{(2)}$ iid $\mathcal{N}(0, I_{2n})$,

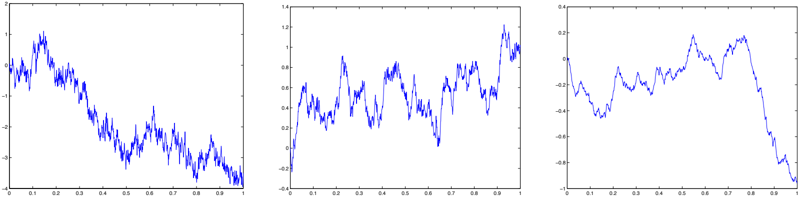
$$R_{2n}[\varepsilon^{(1)} + i\varepsilon^{(2)}] = Z^{(1)} + iZ^{(2)},$$

with $Z^{(1)}, Z^{(2)}$ iid $\mathcal{N}(0, S)$ using $R_{2n} R_{2n}^* = S$. It follows that

$$Y \overset{d}{=} \left(Z_k^{(1)}\right)_{0 \leq k \leq n} \overset{d}{=} \left(Z_k^{(2)}\right)_{0 \leq k \leq n} \sim \mathcal{N}(0, K_Y).$$

Cost $O(n \log(n))$ for $n = 2^p$ to compare with $O(n^3)$ for Choleski method.

Simulation of $(B_H(k/n))_{0 \leq k \leq n}$ for $n = 2^{12}$



$H = 0.3$ $\quad$ $H = 0.5$ $\quad$ $H = 0.7$

- H a.s. critical Hölder exponent : $\forall t, s \in [0,1]$,

$$|B_H(t) - B_H(s)| \leq C|t-s|^{\alpha},$$

for all $\alpha < H$ and not for $\alpha > H$ a.s.

- H a.s. fractal dimension :

$$\dim_{\mathcal{H}} \left(\{(t, B_H(t)), t \in [0,1]\}\right) = 2 - H \text{ a.s.}$$

Extension for exact simulation to 2d Gaussian fields

➥ When stationary and $\mathrm{Cov}(Y_{k_1+l_1,k_2+l_2}, Y_{l_1,l_2}) = r_{k_1,k_2}$ use a block Toeplitz covariance matrix with Toeplitz block and embed with a block circulant matrix [Chan, Wood, 1994, Dietrich, Newsam, 1997]

➥ When only stationary increments simulate the increments but the initial conditions are correlated [Kaplan, Kuo, 1996]

➥ For the fBf approximate by a stationary field with compactly supported covariance function for which the circulant embedding matrix algorithm is running [Stein, 2002, Gneiting et al, 2006]

➥ Conditional simulation procedure when conditional covariances are known [Emery, Lantuejoul, 2006, Brouste et al, 2007]

Turning band method [Matheron, 1973]

When Y is a centered stationary process with covariance $K_Y(t,s) = c_Y(t-s)$ and $U \sim \mathcal{U}(S^1)$ define the field

$$Z(x) = Y(x \cdot U) \text{ for } x \in \mathbb{R}^2$$

such that with $u(\theta) = (\cos(\theta), \sin(\theta))$,

$$c_Z(x) = \mathrm{Cov}(Z(x+y), Z(y)) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} c_Y(x \cdot u(\theta)) d\theta.$$

Then Z is a centered stationary isotropic field (not Gaussian). Defining for $\theta_1, \dots, \theta_K \in [-\pi/2, \pi/2]$ and $\lambda_1, \dots, \lambda_K \in \mathbb{R}^+$

$$Z_K(x) = \sum_{i=1}^{K} \sqrt{\lambda_i} Y^{(i)}(x \cdot u(\theta_i)),$$

with $Y^{(1)}, \dots, Y^{(K)}$ independent realizations of Y the field Z_K is a centered stationary field with covariance

$$c_{Z_K}(x) = \sum_{i=1}^{K} \lambda_i c_Y(x \cdot u(\theta_i)).$$

Anisotropic self-similar random fields

Let $H \in (0,1)$, μ a finite positive measure on S^{d-1}, and $X_{H,\mu} = (X_{H,\mu}(x))_{x\in\mathbb{R}^d}$ a centered Gaussian random field with stationary increments and variogram

$$v_{H,\mu}(x) = \int_{S^{d-1}} |x\cdot\theta|^{2H}\mu(d\theta) = C_{H,\mu}\left(\frac{x}{\|x\|}\right)\|x\|^{2H}.$$

Main Properties :

- H self-similarity : $\forall\lambda > 0$, $X_{H,\mu}(\lambda\cdot) \overset{fdd}{=} \lambda^H X_{H,\mu}(\cdot)$
- H a.s. critical Hölder exponent
- H a.s. fractal dimension :

$$\dim_{\mathcal{H}}\left(\{(t, X_{H,\mu}(t)), t\in[0,1]^d\}\right) = d+1-H \text{ a.s.}$$

- when $\mu(d\theta) = d\theta$, $v_{H,\mu}\circ R = v_{H,\mu}$ for all rotation R and $X_{H,\mu}$ isotropic called (Lévy) fractional Brownian field

Turning band method

When μ_K is a discrete measure ie $\mu_K = \sum_{i=1}^{K} \lambda_i \delta_{\theta_i}$ for some $\theta_1, \dots, \theta_K \in S^{d-1}$ and $\lambda_1, \dots, \lambda_K \in \mathbb{R}^+$,

$$v_{H,\mu_K}(x) = \sum_{i=1}^{K} \lambda_i |x \cdot \theta_i|^{2H} = \sum_{i=1}^{K} \lambda_i \mathrm{Var}(B_H(x \cdot \theta_i))$$

Let $(B_H^{(i)})_{1 \leq i \leq K}$ independent realizations of 1d H-fBm. Then

$$X_{H,\mu_K}(x) := \sum_{i=1}^{K} \sqrt{\lambda_i} B_H^{(i)}(x \cdot \theta_i), \forall x \in \mathbb{R}^d.$$

is a centered Gaussian random field with stationary increments and variogram v_{H,μ_K}.

For $\mu(d\theta) = c(\theta) d\theta$, Riemann approximation for convenient μ_K yields error bound between X_{H,μ_K} and $X_{H,\mu}$.
[HB, Moisan, Richard, J. Comput. Graph. Stat., 15]

Simulation for $d = 2$: choice of lines and weights

To simulate $\left(X_{H,\mu_K}\left(\frac{k}{n},\frac{l}{n}\right)\right)_{0\le k,l\le n}$ one has to simulate for $1 \le i \le K$,

$$B_H^{(i)}\left(\frac{k}{n}\cos(\theta_i) + \frac{l}{n}\sin(\theta_i)\right) \text{ for } 0 \le k,l \le n.$$

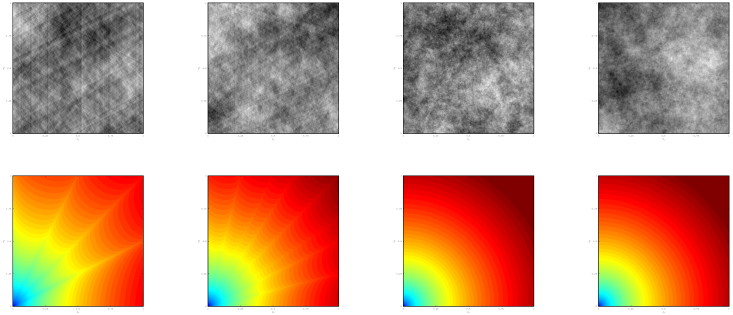
When $\cos(\theta_i) \neq 0$, choose θ_i with $\tan(\theta_i) = \frac{p_i}{q_i}$ for $p_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ s.t.

$$\left(B_H^{(i)}\left(\frac{k}{n}\cos(\theta_i) + \frac{l}{n}\sin(\theta_i)\right)\right)_{k,l} \stackrel{fdd}{=} \left(\frac{\cos(\theta_i)}{nq_i}\right)^H \left(B_H^{(i)}(kq_i + lp_i)\right)_{k,l}.$$

Cost : $O(n(|p_i| + q_i)\log(n(|p_i| + q_i)))$

- Choice of (θ_i) that minimizes the cost via dynamic programming.
- Choice of (λ_i) to get explicit error bounds between X_{H,μ_K} and $X_{H,\mu}$ (Rectangle rule $O(K^{-\min(2H,1)})$ or Trapezoidal rule $O(K^{-\min(2H,1)-1})$)

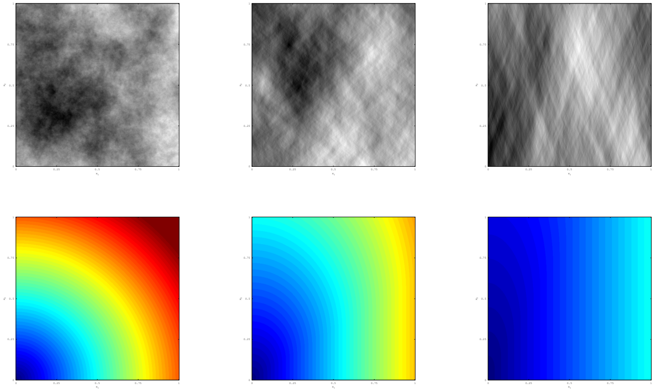
Number of lines



$K = 7$ $\quad$ $K = 15$ $\quad$ $K = 51$ $\quad$ $K = 103$

- Top : realizations of X_{H,μ_K} with $H = 0.2$ and $\mu(d\theta) = d\theta$ for $n = 512$
- Bottom : corresponding v_{H,μ_K}

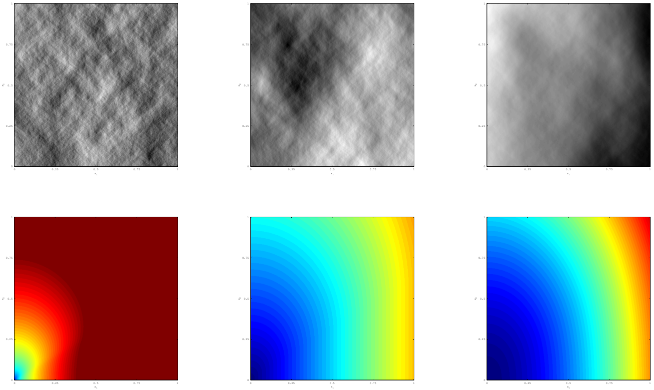
Anisotropy



$\alpha = \pi/2$ $\alpha = \pi/3$ $\alpha = \pi/6$

- Top : realizations of X_{H,μ_K} for $K = 5900$ with $H = 0.5$ and $\mu(d\theta) = \mathbf{1}_{(-\alpha,\alpha)}(\theta)d\theta$ for $n = 512$.
- Bottom : corresponding $v_{H,\mu}$.

Anisotropy



$H = 0.2$ $H = 0.5$ $H = 0.8$

- Top : realizations of X_{H,μ_K} for $K = 5900$ with $\alpha = \pi/3$ and $\mu(d\theta) = \mathbf{1}_{(-\alpha,\alpha)}(\theta)d\theta$ for $n = 512$.
- Bottom : corresponding $v_{H,\mu}$.

Operator scaling random fields

Let $X_{H,E} = (X_{H,E}(x))_{x\in\mathbb{R}^d}$ be a centered Gaussian random field with stationary increments and variogram

$$v_{H,E}(x) = \tau_E(x)^{2H} = \left(\sum_{i=1}^{d} |\langle x,\theta_i\rangle|^{2\alpha_i}\right)^H .$$

Main Properties :

- (E,H) operator scaling property : $\forall\lambda>0,\ X_{H,E}(\lambda^E\cdot) \overset{fdd}{=} \lambda^H X_{H,E}(\cdot)$
- $H\min_{1\leq i\leq d}\alpha_i$ a.s. critical Hölder exponent ; $H\alpha_i$ a.s. critical Hölder exponent in direction $\tilde{\theta}_i$ with $E\tilde{\theta}_i = \alpha_i^{-1}\tilde{\theta}_i$.
- $H\min_{1\leq i\leq d} a_i$ a.s. fractal dimension :

$$\dim_{\mathcal{H}}\left(\{(t,X_{H,\mu}(t)),t\in[0,1]^d\}\right) = d+1-H\min_{1\leq i\leq d} a_i \text{ a.s.}$$

- when $\alpha_1=\ldots=\alpha_d=\alpha$, $X_{H,E}$ is αH self-similar ; isotropic (Lévy) fractional Brownian field iff $\alpha=1$

Fast and exact synthesis for $d = 2$ and $E = diag(\alpha_1^{-1}, \alpha_2^{-1})$

Let us choose $\tau_E(x)^2 := |x_1|^{2\alpha_1} + |x_2|^{2\alpha_2}$ and define for $c_H = 1 - H$,

$$K_{H,E}(x) = \begin{cases} c_H - \tau_E(x)^{2H} + (1 - c_H)\tau_E(x)^2 & \text{if } \tau_E(x) \leq 1 \\ 0 & \text{else} \end{cases}$$

Assume that $K_{H,E}$ is a covariance function on $\mathbb{R}^2$ and define $Y_{H,E}$ a centered Gaussian stationary random field with covariance $K_{H,E}$. Then,

$$\left\{X_{H,E}(x); x \in [0, M]^2\right\}$$

$$\overset{fdd}{=} \left\{Y_{H,E}(x) - Y_{H,E}(0) + \sqrt{1 - c_H}B^{(1)}_{\alpha_1}(x_1) + \sqrt{1 - c_H}B^{(2)}_{\alpha_2}(x_2); x \in [0, M]^2\right\},$$

for $M = \min\left\{0 \leq r \leq 1\, ;\, r^{2\alpha_1} + r^{2\alpha_2} \leq 1\right\}$ and $B^{(1)}_{\alpha_1}$, $B^{(2)}_{\alpha_2}$ two standard independent 1D fractional Brownian motions. [Joint work with C. Lacaux, in preparation]

Fast and exact synthesis for $d = 2$ and E diagonal

If $K_{H,E}$ is a covariance function, since compact support $\subset [-1,1]^2$,

$$K_{H,E}^{per}(x) = \sum_{k \in \mathbb{Z}^2} K_{H,E}(x + 2k),$$

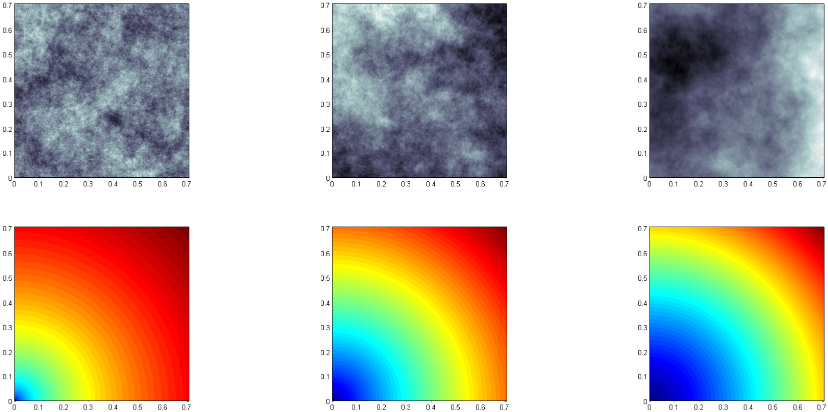
is a periodic covariance function on $\mathbb{R}^2$. Then, since E diagonal, $K_{H,E}(x) = K_{H,E}(|x_1|, |x_2|)$ and $\left(Y_{H,E}^{per}\left(\frac{k}{n}, \frac{l}{n}\right)\right)_{0 \leq k,l \leq 2n}$ has a block circulant covariance matrix diagonalized by 2D discrete Fourier transform.

- ➥ Fast and exact synthesis of $\left(Y_{H,E}\left(\frac{k}{n}, \frac{l}{n}\right)\right)_{0 \leq k,l \leq n}$
  Cost : $O(n^2 \log(n))$.
- ➥ Numerical check for covariance matrix (positivity of eigenvalues).

Isotropic case $E = I_2$

Theorem : [Stein, 02] for $H \in (0, 3/4)$, K_{H,I_2} is a covariance matrix.

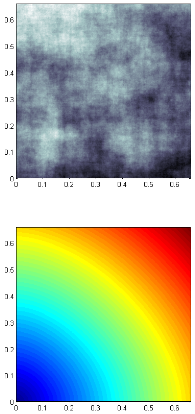
Remark : for $\alpha_i = 1$, one has $B^{(i)}_{\alpha_i}(x_i) = x_i N$ with $N \sim \mathcal{N}(0,1)$.



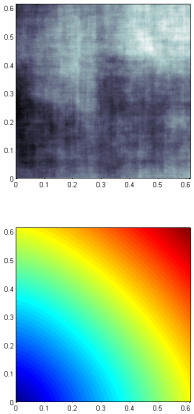
$H = 0.2$ $H = 0.4$ $H = 0.6$

Fast and exact synthesis for $d = 2$ and E diagonal

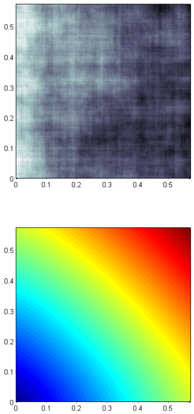
$$H\alpha_1 = H\alpha_2 = 0.5$$



$H = 0.6$



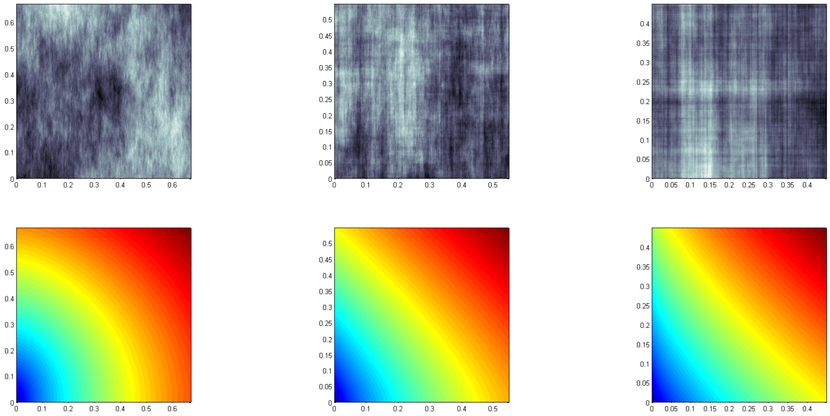
$H = 0.7$



$H = 0.8$

Fast and exact synthesis for $d = 2$ and E diagonal

$H\alpha_1 = 0.3$ and $H\alpha_2 = 0.4$



$H = 0.4$ $H = 0.6$ $H = 0.8$

Estimation based on variogram : 1d case

Let us consider increments of B_H with step u

$$\Delta_u B_H(k) = B_H(k+u) - B_H(k)$$

The sequence $(\Delta_u B_H(k))_{k\in\mathbb{Z}}$ is Gaussian, stationary, centered, with variance

$$v_H(u) = c_H |u|^{2H}$$

$$V_n(u) = \frac{1}{n-u}\sum_{k=0}^{n-1-u} \Delta_u B_H(k)^2 \underset{n\to+\infty}{\longrightarrow} v_H(u) \text{ p.s.}$$

$$\boxed{\widehat{H}_n = \frac{1}{2}\log\left(\frac{V_n(u)}{V_n(v)}\right) / \log\left(\frac{u}{v}\right)}$$

Asymptotic normality ? First step CLT for

$$\frac{V_n(u)}{v_H(u)} = \frac{1}{n-u}\sum_{k=0}^{n-1-u} X_u(k)^2 \text{ with } X_u(k) = \frac{\Delta_u B_H(k)}{\sqrt{v_H(u)}}$$

Asymptotic normality

$$Q_n(u) = \sqrt{n-u}\left(\frac{V_n(u)}{v_H(u)} - 1\right) = \frac{1}{\sqrt{n-u}} \sum_{k=0}^{n-1-u} H_2\left(X_u(k)\right),$$

for $H_2(x) = x^2 - 1$ Hermite polynomial of order 2 and $X_u(k) = \frac{\Delta_u B_H(k)}{\sqrt{v_H(u)}}$ centered stationary Gaussian sequence with UNIT variance and covariance

$$\rho_u(k) = \mathbb{E}\left(X_u(k+l)X_u(l)\right) = O_{|k|\to+\infty}(|k|^{-2(1-H)}).$$

[Breuer Major 83] If $\sigma_u^2 = \sum_{k\in\mathbb{Z}} \rho_u(k)^2 < +\infty$ then

i) $\mathrm{Var}(Q_n(u)) \to 2\sigma_u^2$

ii) $\frac{Q_n(u)}{\sqrt{\mathrm{Var}(Q_n(u))}} \to N$, with $N \sim \mathcal{N}(0,1)$

➥ Asymptotic normality of $Q_n(u)$ for $H < 3/4$

Generalized quadratic variations [Istas Lang 97]

We replace $\Delta_u B_H(k)$ by

$$\Delta_u^{(2)} B_H(k) = B_H(k+2u) - 2B_H(k+u) + B_H(k)$$

such that $\mathrm{Var}(\Delta_u^{(2)} B_H(k)) = c_H^{(2)} |u|^{2H}$ BUT $\rho_u(k) = O(|k|^{-2(2-H)})$

➥ Asymptotic normality for $Q_n(u)$ for all $H \in (0,1)$

Remark

One can also prove that

$$d_{Kol}\left(\frac{Q_n(u)}{\sqrt{Var(Q_n(u))}}, N\right) = \sup_{z\in\mathbb{R}} \left| \mathbb{P}\left(\frac{Q_n(u)}{\sqrt{Var(Q_n(u))}} \leq z\right) - \mathbb{P}(N \leq z)\right| \asymp n^{-1/2}$$

[HB, A. Bonami, I. Nourdin & G. Peccati, Alea (2012)]

Vectorial CLT and triangular array

[Peccati Tudor 04] : $\mathrm{Cov}(Q_n(u), Q_n(v)) \to \sigma_{uv}$ implies that

$$(Q_n(u), Q_n(v)) \to \mathcal{N}\left(0, \begin{pmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{pmatrix}\right).$$

➥ Asymptotic normality of $\widehat{H}_n$ by δ-method

Triangular array - infill estimation : By self-similarity one can replace $\Delta_u^{(2)} B_H(k)$ by

$$\Delta_{u/n}^{(2)} B_H(k/n) = B_H\left(\frac{k+2u}{n}\right) - 2B_H\left(\frac{k+u}{n}\right) + B_H\left(\frac{k}{n}\right)$$

We deduce that

➥ $\widehat{H}_n$ strongly consistent with asymptotic normality

Robustness

More generally, one can replace B_H by Y centered Gaussian process with stationary increments such that

$$v_Y(u) = \mathbb{E}\left(\left(Y(t+u) - Y(t)\right)^2\right) = c_Y |u|^{2H} + O_{|u|\to 0}\left(|u|^{2H+\varepsilon}\right),$$

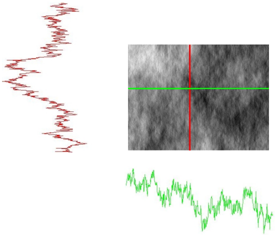
➥ $\widehat{H}_n$ is strongly consistent with asymptotic normality if $\varepsilon > 1/2$

[HB, A. Bonami & J. R. León, *Electron. J. Probab.* (2011)]

Application to fields : line processes

Let us consider $(X(x))_{x\in\mathbb{R}^2}$ centered s.i. Gaussian random field
$\forall\theta \in S^1, x_0 \in \mathbb{R}^2$, the line process $L_{x_0,\theta}(X) = \{X(x_0 + t\theta); t \in \mathbb{R}\}$ is a centered s.i. Gaussian random with variogram

$$v_\theta(t) = \mathbb{E}\left(\left(X(x_0 + t\theta) - X(x_0)\right)^2\right) = v_X(t\theta).$$



Application to fields : line processes

Self-similarity : $\forall c > 0,\ X(c\cdot) \overset{fdd}{=} c^H X(\cdot)$, implies that

$$v_\theta(t) = c_\theta |t|^{2H}.$$

Isotropy : $\forall R \in \mathcal{O}_2(\mathbb{R}),\ X(R\cdot) \overset{fdd}{=} X(\cdot)$, implies that

$$v_\theta = v_{\theta'} \text{ for all } \theta' \in S^1.$$

Operator Scaling Property : $\forall \lambda > 0,\ X(c^E\cdot) \overset{fdd}{=} c^H X(\cdot)$, implies that

$$v_{\tilde{\theta}_1}(t) = c_1 |t|^{2H\alpha_1} \text{ and } v_{\tilde{\theta}_2}(t) = c_2 |t|^{2H\alpha_2},$$

for $E\tilde{\theta}_1 = \alpha_1^{-1}\tilde{\theta}_1$ and $E\tilde{\theta}_2 = \alpha_2^{-1}\tilde{\theta}_2$.

Estimation for OSSGF

Estimation using quadratic variations of order 2 with $N = 2^{10}$ on 100 realizations.

	$H = 0.7$	$H = 0.8$	$H = 0.9$	$H = 1$
	$H1 = 0.7$ and $H2 = 0.7$			
$\hat{H}1$	0.6997 ± 0.0022	0.6990 ± 0.0047	0.7014 ± 0.0167	0.7079 ± 0.0365
$\hat{H}2$	0.7001 ± 0.0021	0.7002 ± 0.0048	0.6991 ± 0.0194	0.7008 ± 0.0344
vmin	$0.2633.10^{-11}$	$0.1651.10^{-11}$	$0.0764.10^{-11}$	
MN	724	689	655	1024
	$H1 = 0.6$ and $H2 = 0.7$			
$\hat{H}1$	0.6002 ± 0.0046	0.5992 ± 0.0103	0.6014 ± 0.0231	0.6034 ± 0.0423
$\hat{H}2$	0.7002 ± 0.0019	0.6995 ± 0.0048	0.7000 ± 0.0157	0.6965 ± 0.0408
vmin	$0.3097.10^{-11}$	$0.2594.10^{-11}$	$0.1411.10^{-11}$	
MN	704	667	633	1024

Estimation for OSSGF

Estimation using quadratic variations of order 2 with $N = 2^{10}$ on 100 realizations.

	$H = 0.4$	$H = 0.6$	$H = 0.8$	$H = 1$
	$H1 = 0.4$ and $H2 = 0.4$			
$\hat{H}1$	0.3999 ± 0.0024	0.4002 ± 0.0037	0.4015 ± 0.0158	0.3952 ± 0.0544
$\hat{H}2$	0.4003 ± 0.0021	0.3996 ± 0.0037	0.4017 ± 0.0147	0.4019 ± 0.0518
vmin	$0.4898.10^{-9}$	$0.3228.10^{-9}$	$0.1671.10^{-9}$	
MN	725	609	512	1025
	$H1 = 0.3$ and $H2 = 0.4$			
$\hat{H}1$	0.3000 ± 0.0043	0.3006 ± 0.0110	0.2971 ± 0.0323	0.2993 ± 0.0449
$\hat{H}2$	0.4001 ± 0.0019	0.3998 ± 0.0035	0.3987 ± 0.0147	0.4007 ± 0.0421
vmin	$0.5627.10^{-9}$	$0.4780.10^{-9}$	$0.2830.10^{-9}$	
MN	687	562	461	1025

References

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