Multicomponent Skyrmion lattices and their excitations

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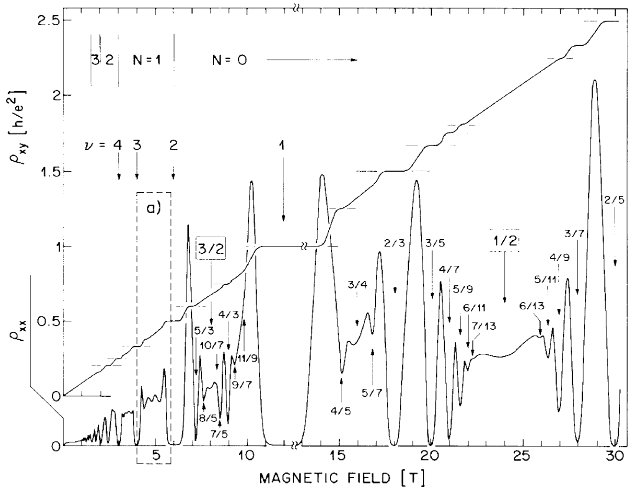
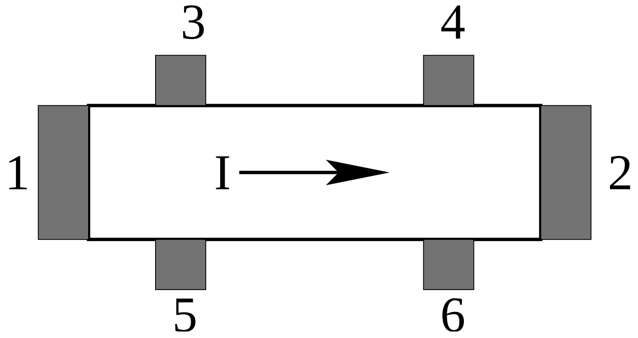
Quantum Hall effect





$$R_{xx} = (V(3) - V(4))/I$$

$$R_{xy} = (V(3) - V(5))/I$$

Quantum nature of Hall resistance plateaus

Plateaus observed for (ν integer):

$$\rho_{xy} = \frac{B}{ne} = \frac{h}{\nu e^2}$$

$\longrightarrow$ Quantized electronic densities:

$$n = \nu \frac{eB}{h}$$

In terms of $\Phi_0 = \frac{h}{e}$: “Flux quantum”

$$N_{\text{electrons}} = \nu \frac{\text{Total magnetic flux}}{\Phi_0}$$

Energy spectrum for a single electron

$$H = \frac{1}{2m}(\mathbf{P} + e\mathbf{A})^2, \quad \mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} \ \text{ spatially uniform.}$$

Define gauge invariant $\mathbf{\Pi} = \mathbf{P} + e\mathbf{A} = m\mathbf{v}$

$\{p_i, r_j\} = \delta_{ij}, \quad i, j \in \{x, y\}, \quad \{\Pi_x, \Pi_y\} = eB$

$\longrightarrow$ Harmonic oscillator spectrum: $E_n = \hbar\omega(n + 1/2)$, $\omega = eB/m$

Conserved quantities (also generators of magnetic translations)

$\mathbf{v} = \omega\hat{\boldsymbol{z}} \wedge (\mathbf{r} - \mathbf{R}), \quad \mathbf{R} = \mathbf{r} + \frac{\hat{\boldsymbol{z}} \wedge \mathbf{\Pi}}{eB}, \quad \{R_x, R_y\} = -\frac{1}{eB}, \quad \{R_i, \Pi_j\} = 0$

Heisenberg principle: $B\Delta R_x \Delta R_y \simeq \frac{h}{e} = \Phi_0$

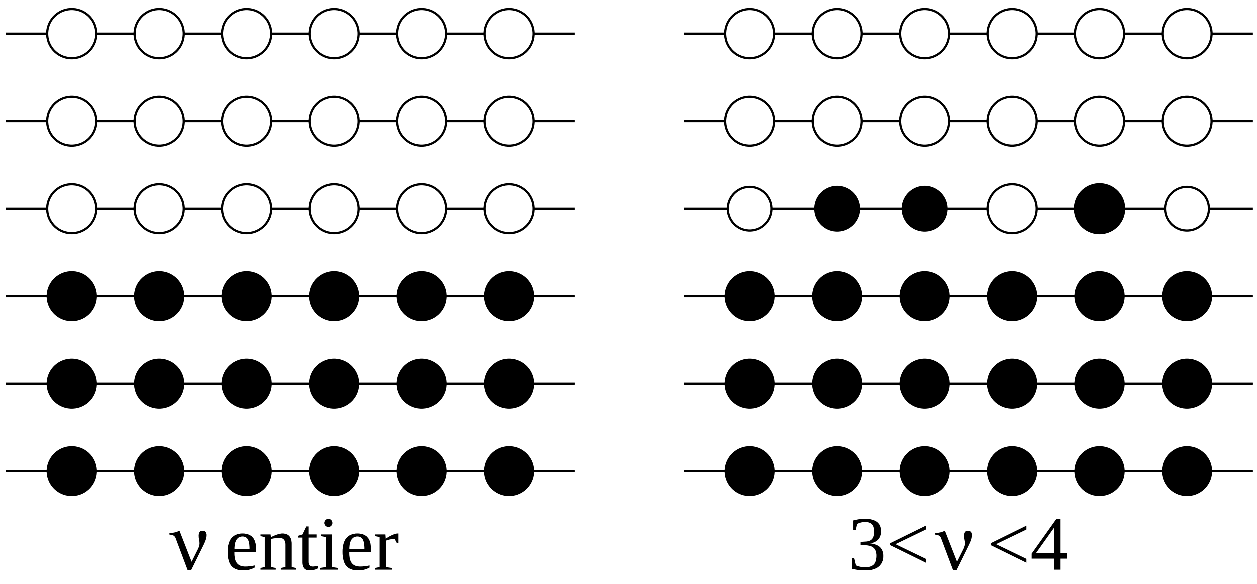
$\longrightarrow$ Magnetic length $l = \sqrt{\frac{\hbar}{eB}}$

Landau levels are degenerate

Intuitively, each state occupies the same area as a flux quantum Φ_0, so that the number of states per Landau level =

$$\frac{\text{Total magnetic flux}}{\Phi_0}$$

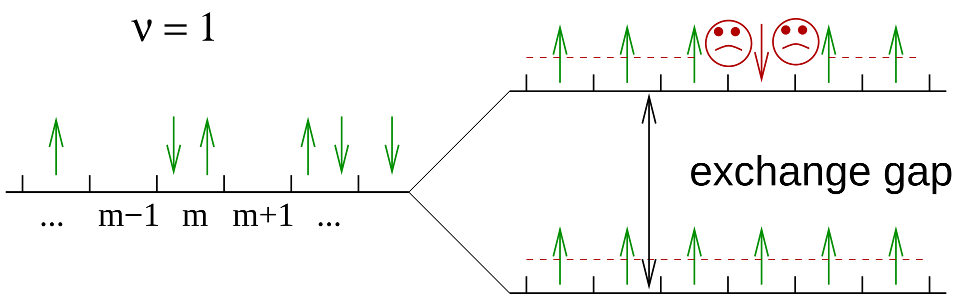
ν is interpreted as the number of occupied Landau levels



Ferromagnetism at $\nu = 1$

Coulomb repulsion favours anti-symmetric orbital wavefunction

$\longrightarrow$ spin wavefunction: symmetric (ferromagnet)



no interactions | with repulsive interactions

in QH systems : no kinetic−energy cost !
(LL ~ flat band)

A class of trial states near $\nu = 1$

Take antisymmetrized products of single particle states (Slater determinants or Hartree-Fock states): $|S_\psi\rangle = \bigwedge_{\alpha=1}^{N} |\Phi_\alpha\rangle$ where $\Phi_{\alpha,a}(r) = \chi_\alpha(r)\psi_a(r)$, $r = (x, y)$, $a \in \{1, ..., d\}$.
$\chi_\alpha(r) \longrightarrow$ electron position.
$\psi_a(r) \longrightarrow$ slowly varying spin background. ($\langle\psi(r)|\psi(r)\rangle = 1$).

In the $d = 2$ case, if σ_a denote Pauli matrices:
Associated classical spin field: $n_a(r) = \langle\psi(r)|\sigma_a|\psi(r)\rangle$
Topological charge: $N_{\rm top} = \frac{1}{4\pi}\int d^{(2)}r\,(\partial_x\vec{n} \wedge \partial_y\vec{n}) \cdot \vec{n}$

Because of large magnetic field, we require that orbital wave-functions $\Phi_{\alpha,a}(r)$ minimize their kinetic energy.

Extra charges at $\nu = 1$ induce Skyrmion textures

Sondhi, Karlhede, Kivelson, Rezayi, PRB **47**, 16419, (1993)

$$\langle \Phi_\alpha | (P - eA)^2 | \Phi_\alpha \rangle = \langle \chi_\alpha | (P - eA_{\text{eff}})^2 + V_{\text{eff}} | \chi_\alpha \rangle$$

$$V_{\text{eff}} = \langle \nabla \psi | \nabla \psi \rangle - \langle \nabla \psi | \psi \rangle \langle \psi | \nabla \psi \rangle$$

$$A_{\text{eff}} = A - \Phi_0 \frac{1}{2\pi} \mathcal{A}$$

Berry connection: $\mathcal{A} = \frac{1}{i} \langle \psi | \nabla \psi \rangle$

Generalized topological charge: $\oint \mathcal{A} . d\mathbf{r} = 2\pi N_{\text{top}}$
(This coincides with the previous notion when $d = 2$).

Extra charges at $\nu = 1$ induce Skyrmion textures

Sondhi, Karlhede, Kivelson, Rezayi, PRB **47**, 16419, (1993)

$$\langle \Phi_\alpha | (P - eA)^2 | \Phi_\alpha \rangle = \langle \chi_\alpha | (P - eA_{\text{eff}})^2 + V_{\text{eff}} | \chi_\alpha \rangle$$

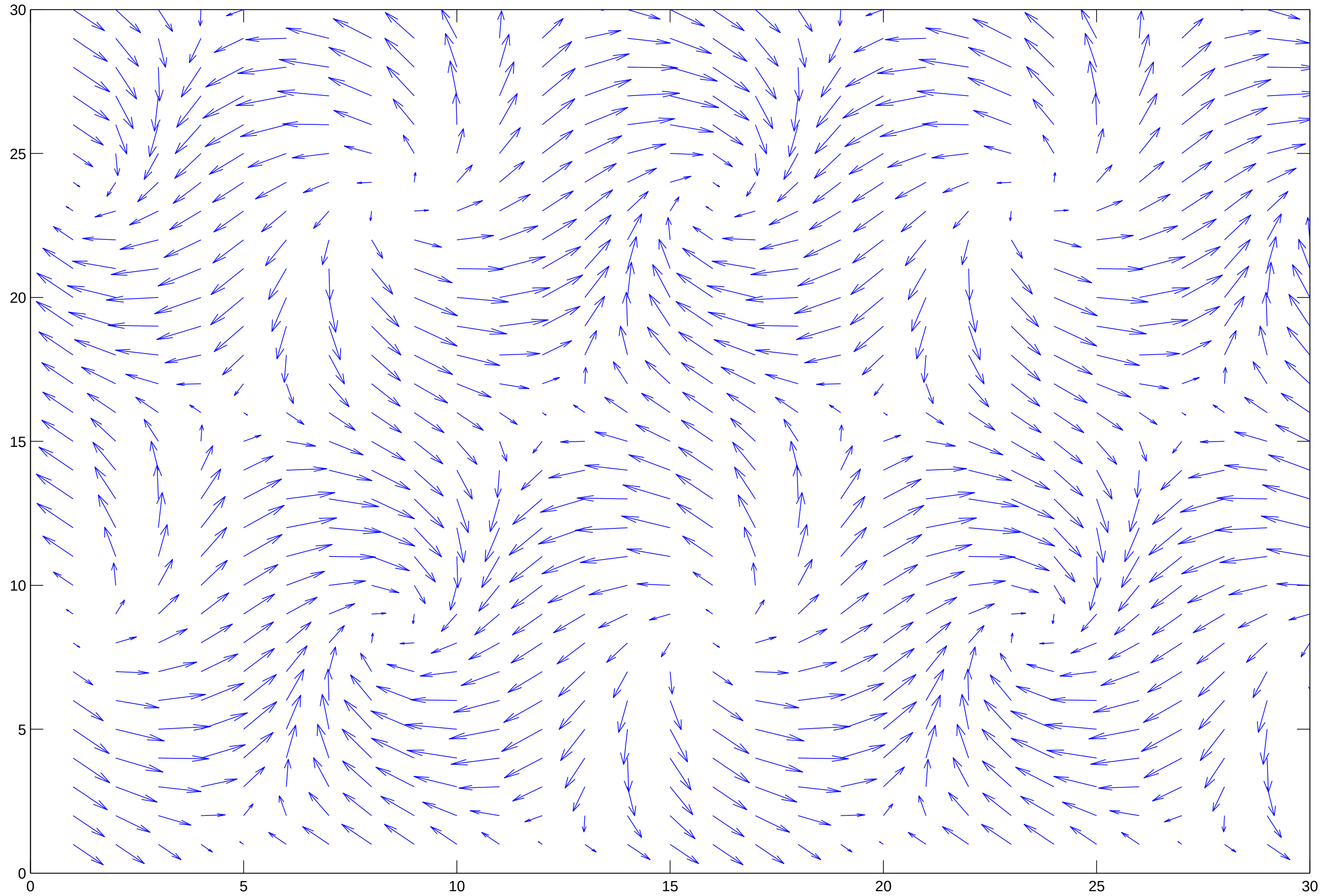
Consequences:
The charge orbitals $\chi_\alpha(r)$ lie in the lowest Landau level of A_{eff}.
There are $N_{\text{eff}} = \text{Effective flux}/\Phi_0$ states in this level.
Condition to minimize Coulomb energy:

$$N_{\text{electrons}} = N_{\text{eff}}$$

Finally:

$$N_{\text{electrons}} = N(\nu = 1) - N_{\text{top}}$$

Picture of a Skyrmion crystal



Skyrmion crystals in electronic systems

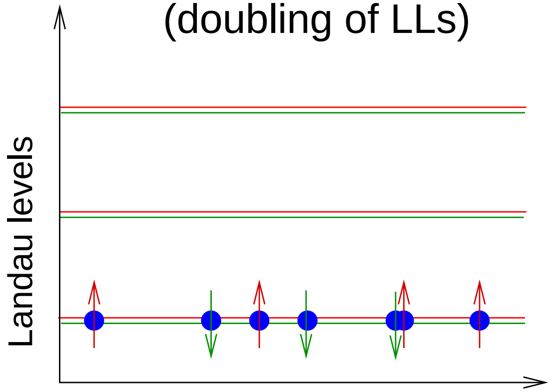
Theoretical prediction: Brey, Fertig, Côté and MacDonald, PRL **75**, 2562 (1995)
Specific heat peak: Bayot et al. PRL **76**, 4584 (1996) and PRL **79**, 1718 (1997)
Increase in NMR relaxation: Gervais et al. PRL **94**, 196803 (2005)
Raman spectroscopy: Gallais et al, PRL **100**, 086806 (2008)
Microwave spectroscopy: Han Zhu et al. PRL **104**, 226801 (2010)

Recent observation (neutron scattering) on the chiral itinerant magnet MnSi: Mühlbauer et al, Science **323**, 915 (2009)

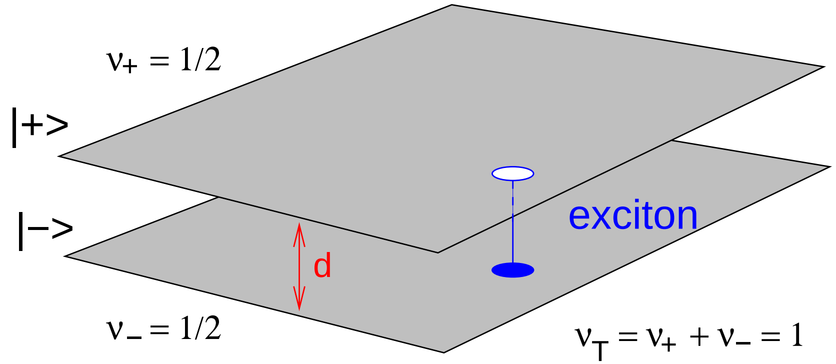
Multi-Component Systems (Internal Degrees of Freedom)

(A) physical spin: SU(2)

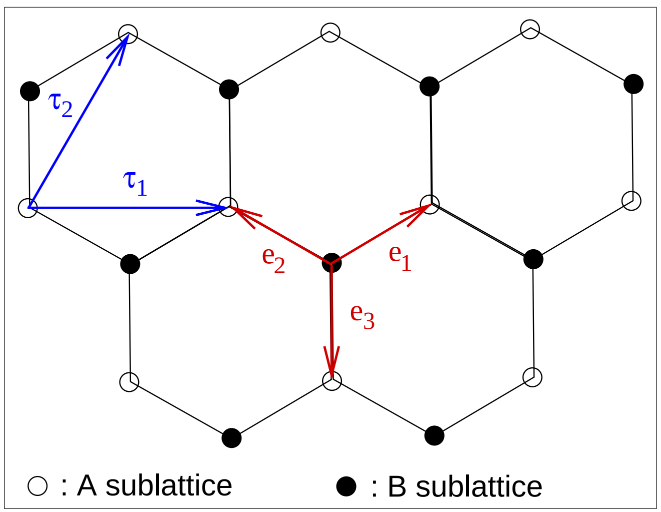
(doubling of LLs)



(B) bilayer: SU(2) isospin



(C) graphene (2D graphite)

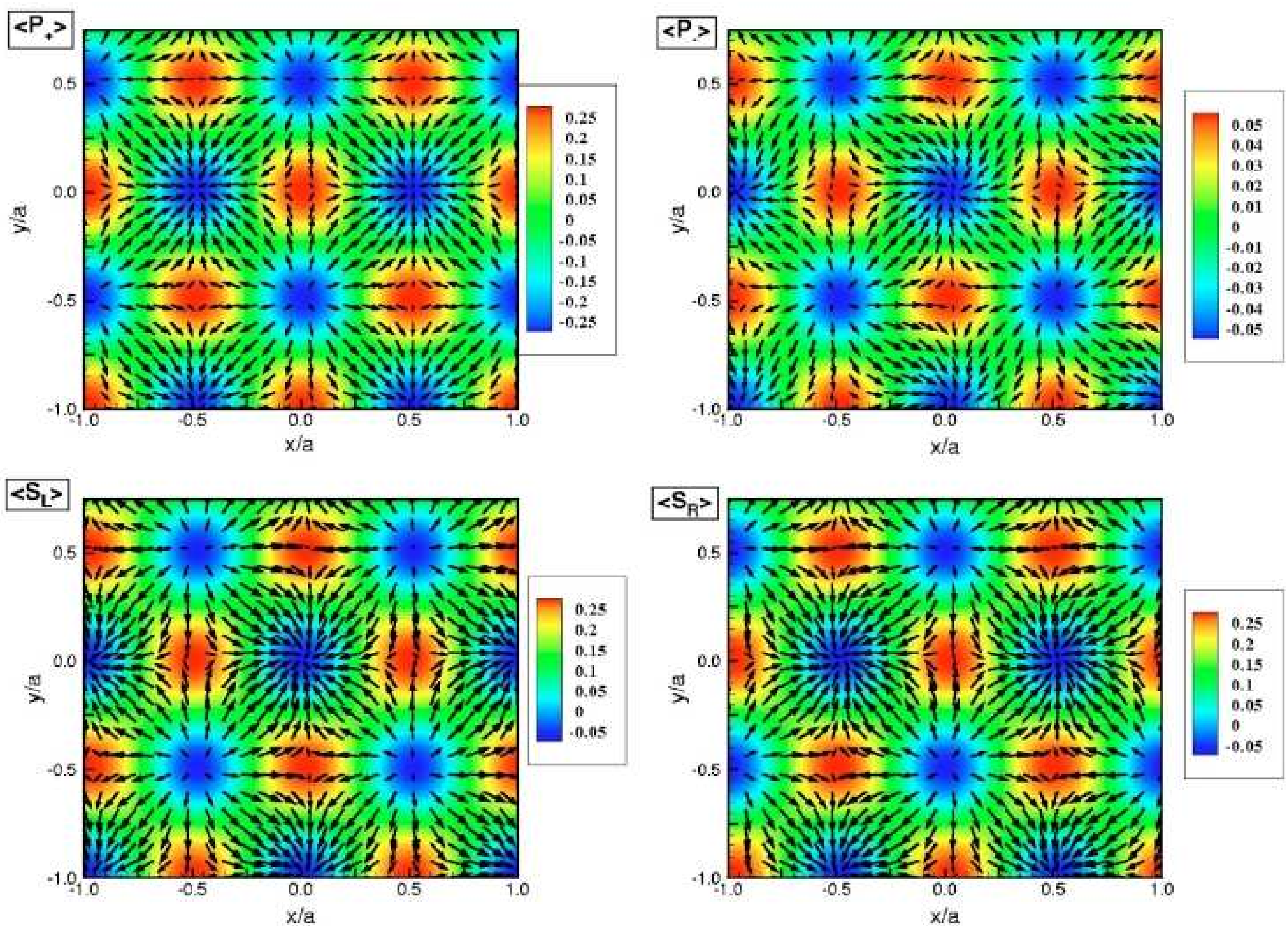


two−fold valley degeneracy

→ SU(2) isospin

spin + isospin : SU(4)

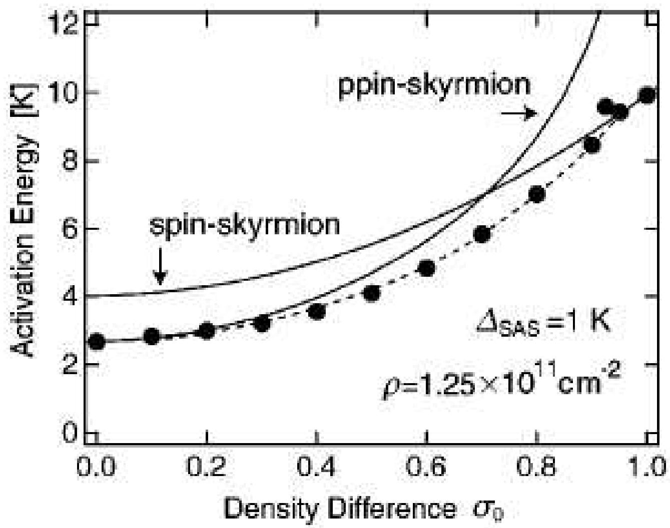
The case for entangled textures (I)



Bourassa et al, Phys. Rev. B **74**, 195320 (2006)

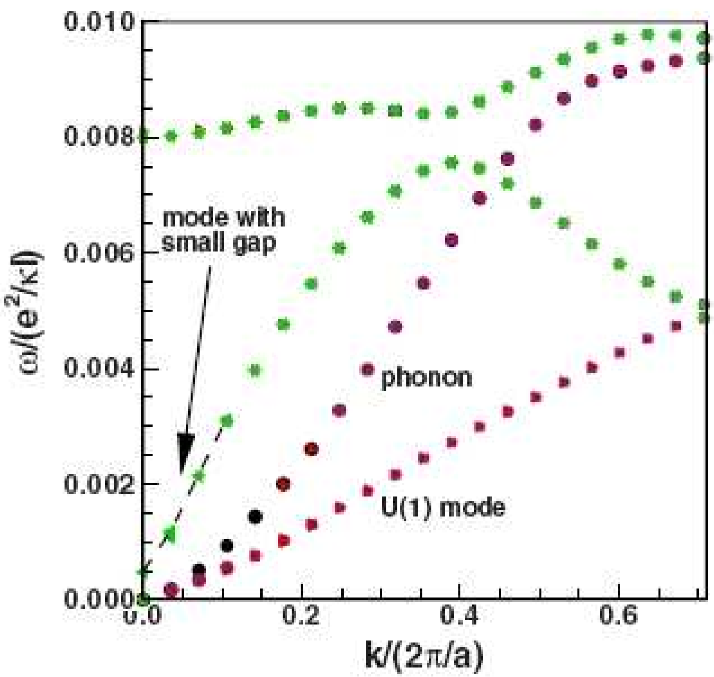
The case for entangled textures (II)

Bilayer with charge imbalance



Ezawa, Tsitsishvili,
Phys. Rev. B **70**, 125304, (2004)

Collective mode spectrum



Côté et al.,
Phys. Rev. B **76**, 125320, (2007)

Enforcing projection onto the lowest Landau level

Problem: in general, factorization of single particle orbitals is not compatible with lying in the L.L.L.
Important exception: holomorphic textures.
Solution: diagonalize an auxiliary Zeeman-like Hamiltonian:
$\hat{H}_Z = -\mathcal{P}_{LLL} \frac{\psi_a(r)\bar{\psi}_b(r)}{\sum_{i=1}^{d} \bar{\psi}_b(r)\psi_b(r)} \mathcal{P}_{LLL}$. In absence of $\mathcal{P}_{LLL}$, this operator has two highly degenerate eigenvalues, 1 and 0.
Effects of $\mathcal{P}_{LLL}$(F. Faure and B. Zhilinskii, (2001)):
Lifts the degeneracy, turning the spectrum of $\hat{H}_Z$ into two bands, separated by a gap.
The dimensions of eigenspaces associated to eigenvalues 1 and 0 are respectively $N - N_{\text{top}}$ and $(d-1)N + N_{\text{top}}$.
The projector $\hat{P}$ associated to the former band can be computed by a semi-classical expansion, the small parameter being the magnetic length l.

Semi-classical expansion of $\hat{P}$

- Start from $[\hat{P}, \hat{H}_Z] = 0$ and $\hat{P}^2 = \hat{P}$.
- Represent operators in the LLL, $\hat{P}$ and $\hat{H}_Z$ by their (anti-Wick) symbols, P and $P_0 = \frac{\psi_a(r)\bar{\psi}_b(r)}{\sum_{i=1}^{d} \bar{\psi}_b(r)\psi_b(r)}$.
- Expand $P = P_0 + l^2 P_1 + l^4 P_2 + \ldots$, and like-wise for star products. First quantum correction: $P_1 = (\mathbf{1} - 2P_0)(P_0 \star_1 P_0)$.
- Form Slater determinant $|\mathcal{S}_\psi\rangle$ from projector $\hat{P}$.
- Transform anti-Wick (contravariant) symbols into Wick (covariant) symbols to get local density matrix $P_{\text{cov}}(r)$ in state $|\mathcal{S}_\psi\rangle$. $P_{\text{cov}} = P_0 + 2l^2 \partial_{\bar{z}} \partial_z P_0 + l^2 P_1 + \mathcal{O}(l^4)$
- Local particle density: $\rho(r) = \frac{1}{2\pi l^2} - Q_{\text{top}}(r)$

$\mathbb{C}P(d-1)$ *model for exchange energy*

d-component spinor field $|\psi(r)\rangle$ parametrizes a Slater determinant $|\mathcal{S}_\psi\rangle$. Consider two-body interactions (Coulomb) and look at first quantum correction in total energy:

$$\mathcal{E}_{ex} = \langle \mathcal{S}_\psi | H_{\rm int} | \mathcal{S}_\psi \rangle = \int d^{(2)}r \left(\frac{\langle \nabla\psi|\nabla\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\nabla\psi|\psi\rangle\langle\psi|\nabla\psi\rangle}{\langle\psi|\psi\rangle^2} \right)$$

Berry connection: $\mathcal{A} = \frac{1}{i}\langle\psi|\nabla\psi\rangle$
Topological charge: $\oint \mathcal{A}.d\mathbf{r} = 2\pi N_{\rm top}$

$$\mathcal{E} \geq \pi |N_{\rm top}|$$

Lower bound is reached when $|\psi(r)\rangle$ is holomorphic ($N_{\rm top} > 0$) or anti-holomorphic: ($N_{\rm top} < 0$), leading to a massive degeneracy.

Slater determinants as coherent states

Variational formulation of Schrödinger equation:

$\delta \int_{t_i}^{t_f} \left(i\langle \Psi | \frac{\partial \Psi}{\partial t} \rangle - \langle \Psi | H | \Psi \rangle \right) dt = 0$

Time-dependent Hartree-Fock equations of motion: constrained dynamics within the manifold $|\Psi(t)\rangle = |\mathcal{S}_{\psi(t)}\rangle$. To lowest order in l^2 expansion:

$$\langle \mathcal{S}_{\psi(t)} | \frac{\partial \mathcal{S}_{\psi(t)}}{\partial t} \rangle = \int \frac{d^2 r}{2\pi l^2} \langle \psi(r,t) | \frac{\partial \psi(r,t)}{\partial t} \rangle + \mathcal{O}(1) \equiv \alpha(\psi(t)) [\frac{\partial \psi(r,t)}{\partial t}]$$

Considering $\omega = -i d\alpha$ allows us to view the set of classical textures as an infinite dimensional symplectic manifold. The subset of holomorphic textures $\mathcal{D}$ is a submanifold of finite dimension. Observation: the restriction of ω to $\mathcal{D}$ is non-degenerate.

Hamiltonians with continuous degeneracies (I)

Normal form for positive Hamiltonians near a degenerate equilibrium point (Williamson):

$$H = \frac{1}{2} \sum_{j=N_0+1}^{N_0+N_d} p_j^2 + \frac{1}{2} \sum_{j=N_0+N_d+1}^{N} \omega_j (p_j^2 + q_j^2)$$

N_0, N_d, and $N_m = N - N_0 - N_d$ are the numbers of zero modes, of drift modes, and of massive modes respectively.

Hamiltonians with continuous degeneracies (II)

Relative Darboux theorem: if a classical Hamiltonian system admits a submanifold $\mathcal{D}$ of degenerate equilibria with a constant Williamson type (N_0,N_d,N_m), there exists locally canonical coordinates, such that:

- $\mathcal{D}$ is defined by:
  $p_{N_0+1} = ... = p_{N_0+N_d} = p_{N_0+N_d+1} = ... = p_N = 0$ and
  $q_{N_0+N_d+1} = ... = q_N = 0$.
- Near $\mathcal{D}$, the previous normal form for H is valid, with ω_j functions of the slow coordinates
  $(p_s, q_s) \equiv (p_1, ..., p_{N_0}, q_1, ..., q_{N_0}, q_{N_0+1}, ..., q_{N_0+N_d})$, and the kinetic term takes the form: $\frac{1}{2}\sum_{j=N_0+1}^{N_0+N_d} A_{ij}(p_s, q_s)p_i p_j$.

Useful special case: if the restriction of ω to $\mathcal{D}$ is non-degenerate, then $N_d = 0$.

Quantum degeneracy among holomorphic textures

Question: how does the quantum ground-state energy of the massive modes depend on the slow variables (p_s, q_s) ?
Toy model: Assume a single particle Hamiltonian ($z = p + iq$) such that $H(z, \bar{z}) \equiv \langle \Phi_{\bar{z}} | \hat{H} | \Phi_{\bar{z}} \rangle$ is minimal at $z = 0$. Then:
$H(z, \bar{z}) = E_0 + \frac{\omega_0}{2} \bar{z} z + \frac{\Delta}{4} z^2 + \frac{\bar{\Delta}}{4} \bar{z}^2 + ...$
Quantum-mechanically: $\hat{H} = E_0 + \hbar\omega_0 \, b^+ b + \frac{\hbar\Delta}{2} (b^+)^2 + \frac{\hbar\bar{\Delta}}{2} b^2 + ...$, with $[b, b^+] = 1$. Its ground-state energy is:
$E_{gs} = E_0 + \frac{\hbar}{2} (\sqrt{\omega_0^2 - \Delta^2} - \omega_0)$. So $E_{gs} = E_0$ if $\Delta = 0$. This holds to all orders in $\hbar$ if the Taylor expansion of the covariant symbol $H(z, \bar{z})$ does not contain any term of the form z^n or $\bar{z}^n$.
Main remark: the $\mathbb{C}P(d-1)$ action, seen as a covariant symbol, has this property, z being replaced by $\{\delta\psi_a(r)\}_{a,r}$, and $\bar{z}$ by $\{\overline{\delta\psi_a(r)}\}_{a,r}$.

Spectrum of the Hessian matrix (I)

Consider small deviations $|\psi\rangle \to |\psi\rangle + \sqrt{\langle\psi|\psi\rangle}|\phi\rangle$ away from holomorphic spinor $|\psi\rangle$.

$$\mathcal{E} = \pi|N_{\rm top}| + 2\langle\phi|M^+PM|\phi\rangle + \ldots$$

$$M|\phi\rangle = |\partial_{\bar z}\phi\rangle + \frac{1}{2}\frac{\langle\partial_{\bar z}\psi|\psi\rangle}{\langle\psi|\psi\rangle}|\phi\rangle$$

$$P|\phi\rangle = |\phi\rangle - \frac{|\psi(z)\rangle\langle\psi(z)|}{\langle\psi(z)|\psi(z)\rangle}|\phi\rangle$$

Key property:

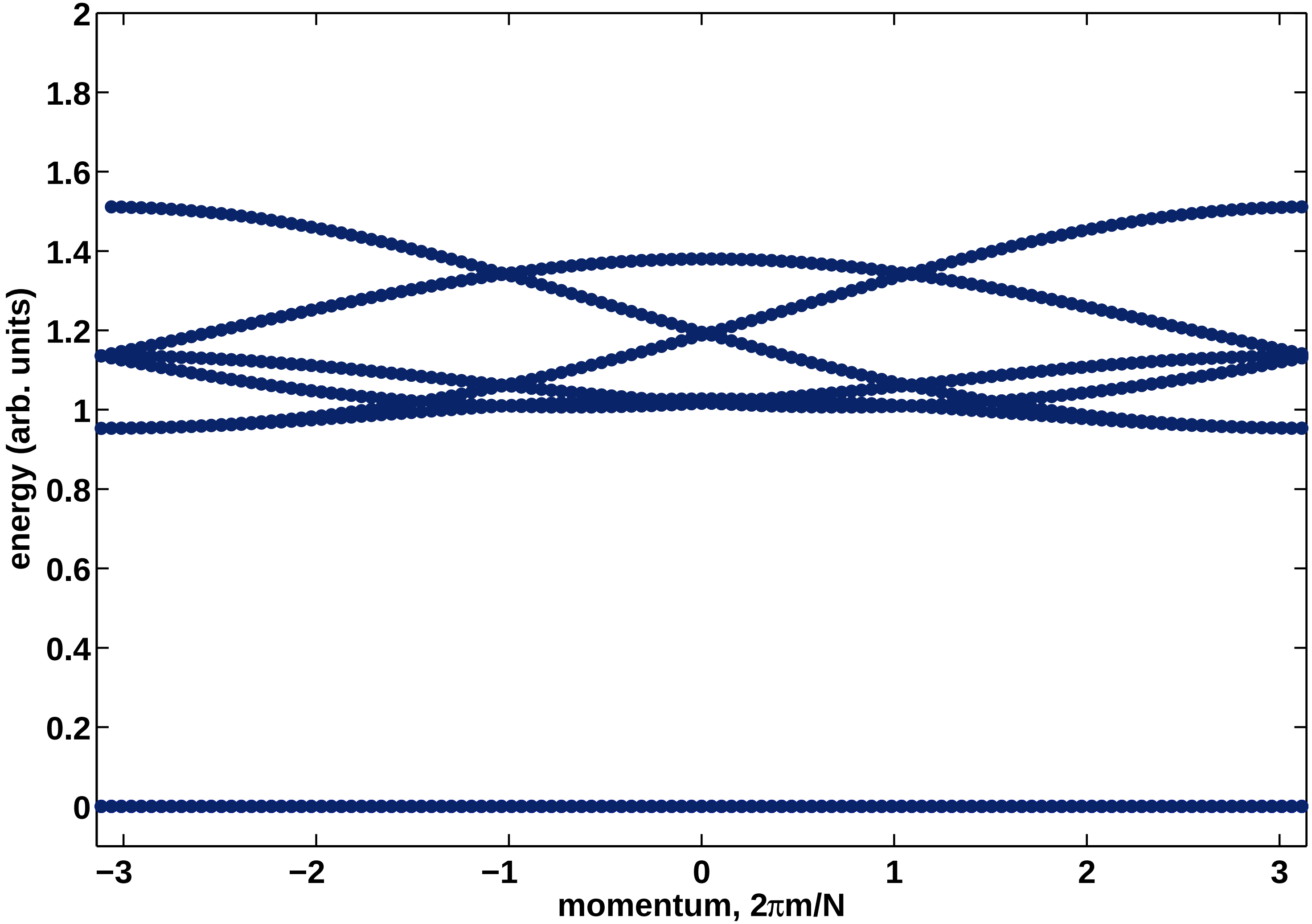
$[M, M^+] = \frac{1}{2}\mathcal{B}(r) = \pi Q(r)$

If $\mathcal{B}(r)$ constant, the spectrum of M^+M is $\{\frac{\mathcal{B}}{2}n, n = 0, 1, 2, \ldots\}$.

At large d, we may expect that the effect of P is small.

Most likely, Hessian of $\mathbb{C}P^{(d-1)}$ model is gapped, with an energy gap of order $\frac{e^2}{4\pi\epsilon l}nl^2$. $\quad$ ($l = \sqrt{\hbar/eB}$, $\overline{Q(r)} = n$).

Spectrum of the Hessian matrix (II)



Variational evaluation of the hessian spectrum for $d = 3$

Variational approach for lattice of textures

$$\mathcal{E} = \mathcal{E}_{ex} + \mathcal{E}_{el}, \quad \mathcal{E}_{el} = \frac{1}{2}\int d^{(2)}r_1 \int d^{(2)}r_2 Q(r_1)u(r_1 - r_2)Q(r_2)$$

$$u(r) = \frac{e^2}{4\pi\epsilon|r|}$$

Assume an average charge density $\overline{Q(r)} = n$, then $\mathcal{E}_{el}/\mathcal{E}_{ex} = l n^{1/2}$, where $l = \sqrt{\hbar/eB}$. In the *dilute limit*, $\mathcal{E}_{ex} \gg \mathcal{E}_{el}$.

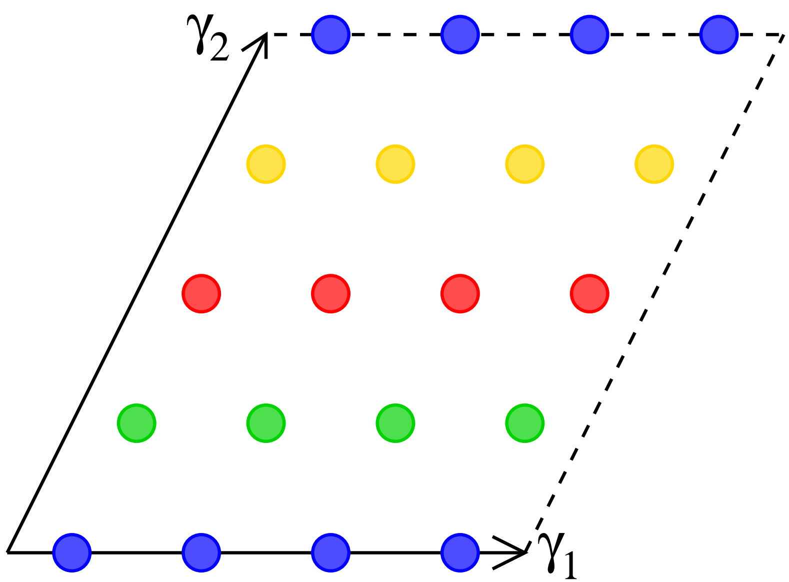
Main approximation: Minimize $\mathcal{E}$ among the configurations that minimize $\mathcal{E}_{ex}$. That is, we look for holomorphic d-component spinor configurations $|\Psi(r)\rangle$ with given $\overline{Q(r)} = n$, such that $\mathcal{E}_{el}$ is minimum.

Physical intuition: One should make $Q(r)$ as homogeneous as possible. In particular, it is natural to consider first periodic patterns.

Periodic textures with lowest energy

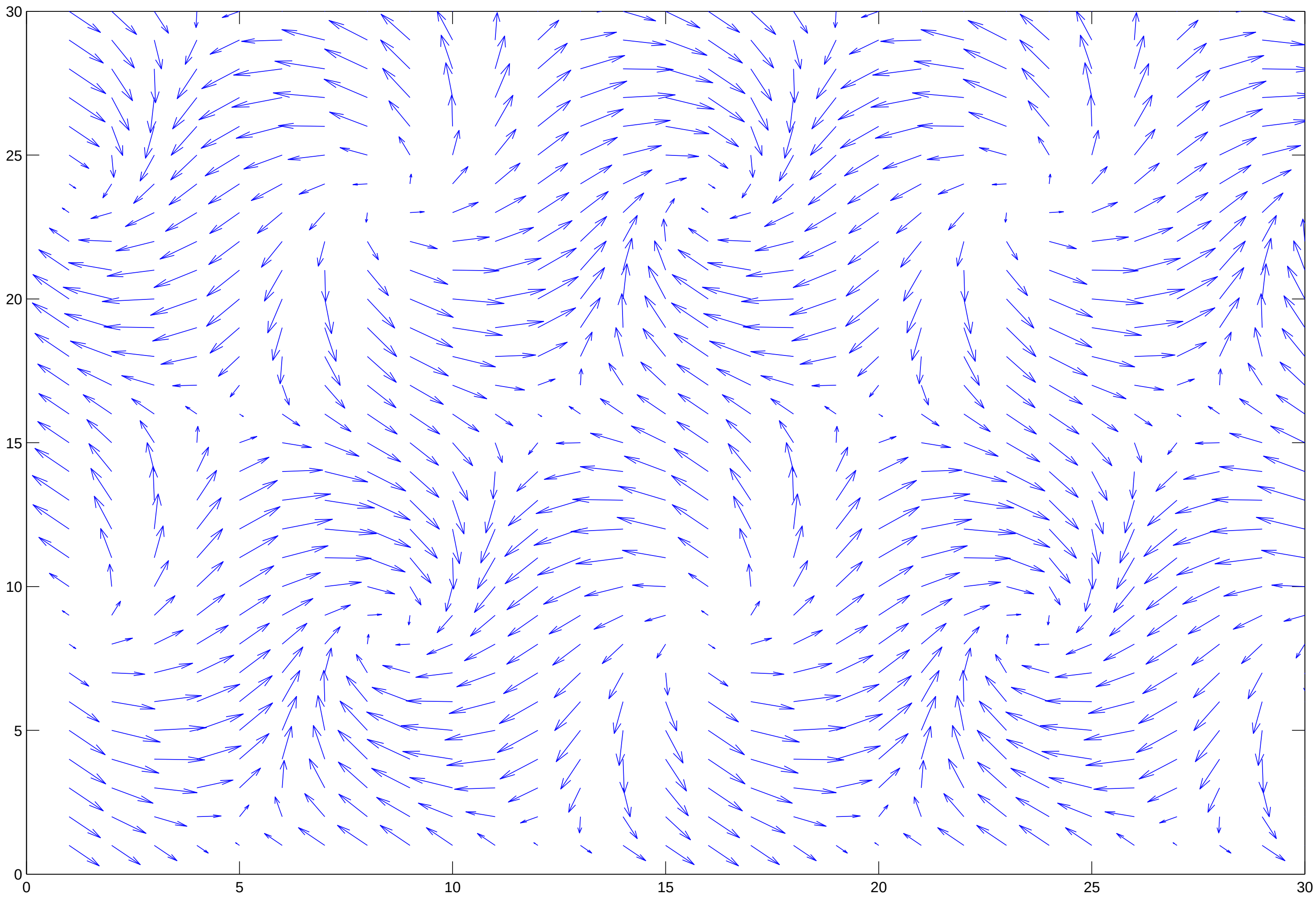
$$|\Psi(z)\rangle = \begin{pmatrix} \theta_0(z) \\ \theta_1(z) \\ \cdot \\ \cdot \\ \cdot \\ \theta_{d-1}(z) \end{pmatrix}$$

Pattern of zeros
(d=4)

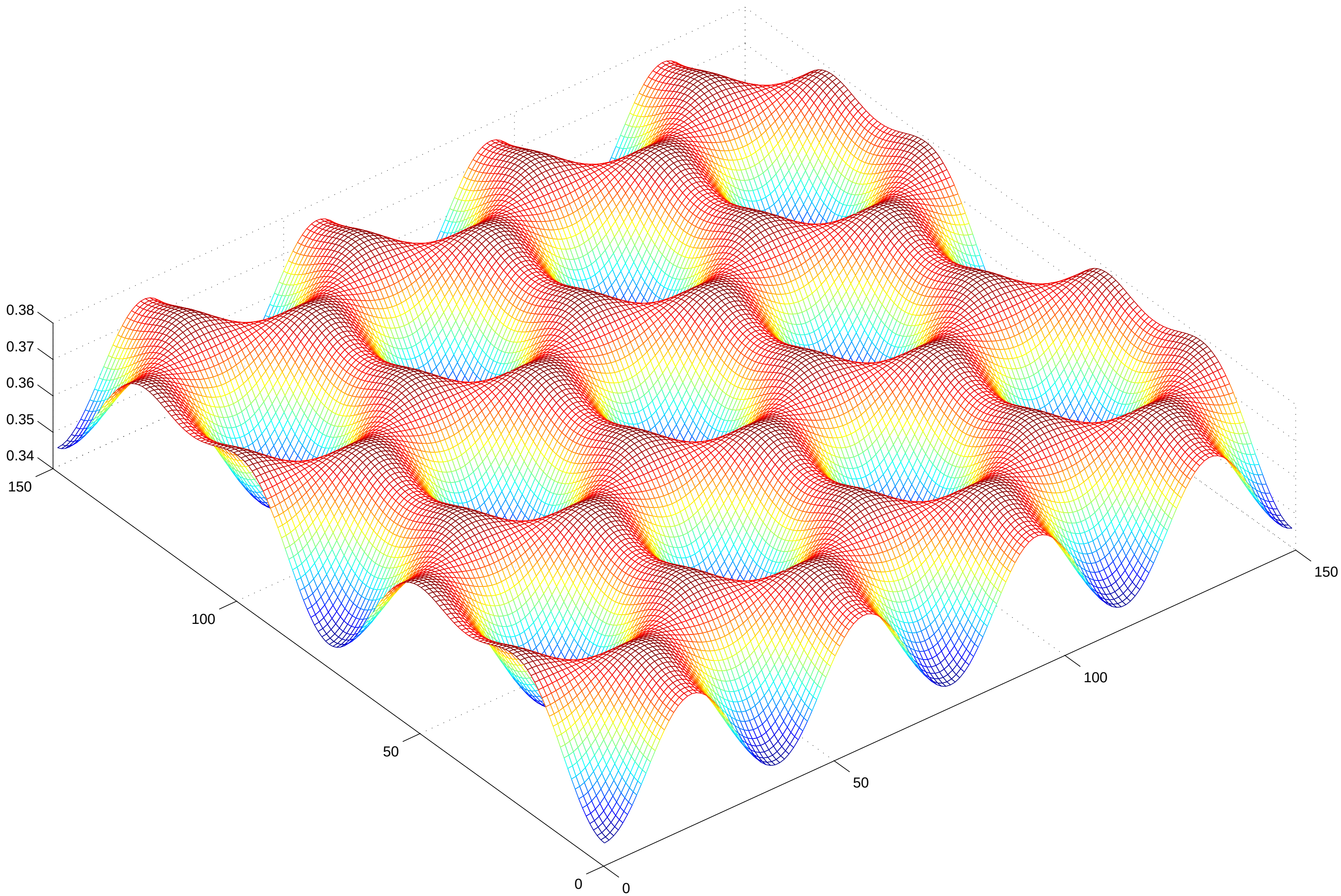


Spontaneously broken $SU(d)$ symmetry : if $g \in SU(d)$, changing $|\Psi(z)\rangle$ into $g\,|\Psi(z)\rangle$ gives another physically inequivalent ground-state.

Periodic texture $d = 2$



Periodic texture $d = 4$



Spatial variations of topological charge

$Q(r)$ is always γ_1/d and γ_2/d periodic.

At large d the modulation contains mostly the lowest harmonic, and its amplitude decays exponentially with d.

Large d behavior for a square lattice:

$$Q(x,y) \simeq \frac{2}{\pi} - 4d e^{-\pi d/2}[\cos(2\sqrt{d}x) - 2e^{-\pi d/2}\cos^2(4\sqrt{d}x) + (x \leftrightarrow y)] + ...$$

Only the triangular lattice seems to yield a true local energy minimum. This is most directly seen by computing eigenfrequencies of small deformation modes.

Consequences of $U(d)$ symmetry

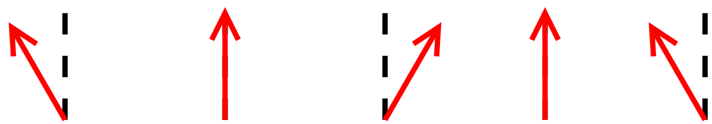
Zero-momentum sector: Hamiltonian system with $N = d^2$ degrees of freedom.

If $g \in U(d)$, the transformation $M \to gM$ preserves equations of motion.

The $U(d)$-orbit of the periodic ground-state has dimension d^2. Furthermore, it is lagrangian.

Example of a system with a degenerate manifold of Williamson type $(N_0, N_d, N_m) = (0, d^2, 0)$.

Collective mode spectrum (I)

Analogy with spin-wave theory: 

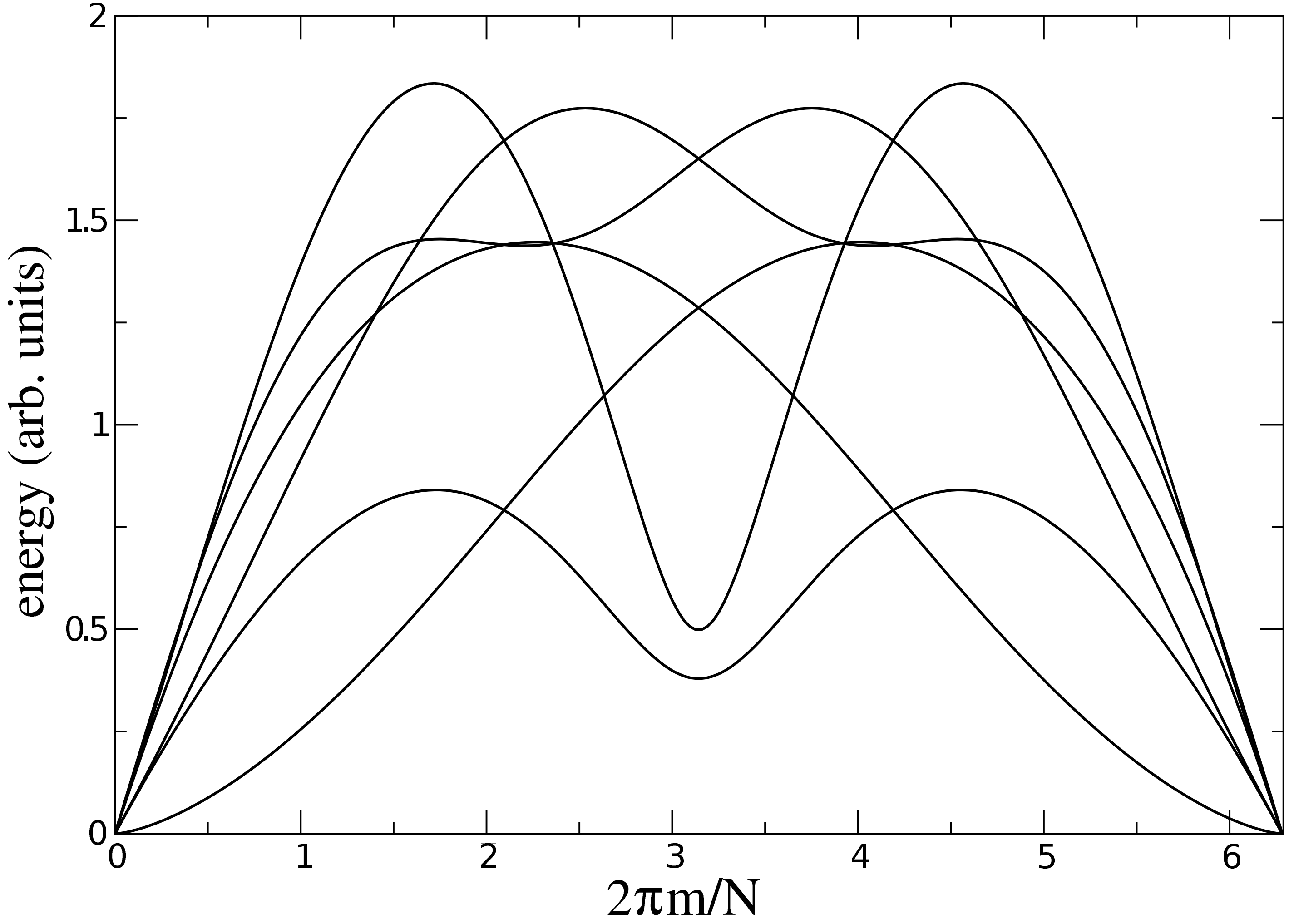
$$\psi_a(r) = (\delta_{ab} + M_{ab}(r))\theta_b(r)$$

$M_{ab}(r)$ gives d^2 degrees of freedom for each *pseudo-momentum*, so there are d^2 branches (positive frequencies) in the excitation spectrum: the situation is reminiscent of a non-collinear antiferromagnet.

Get one *magnetophonon* with $\omega \simeq k^{1+\alpha/2}$ if $u(r) \simeq r^{-\alpha}$, and $d^2 - 1$ *spin-waves* with linear dispersion.

Collective mode spectrum (II)

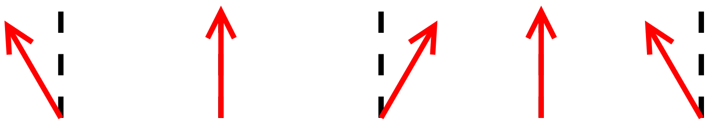
Numerical spectrum for $d = 3$ and Coulomb interactions



D. Kovrizhin, B. D. and R. Moessner, Phys. Rev. Lett. **110**, 186802, (2013)

An $U(d)$ σ-model for collective dynamics? (I)

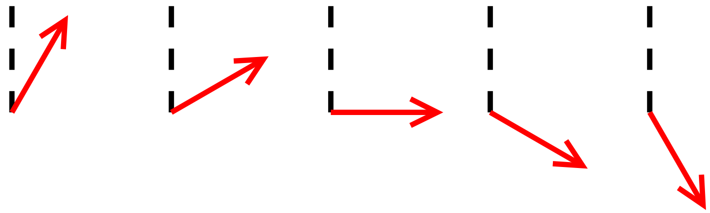
Linear spin-waves



$\psi_a(r) = (\delta_{ab} + M_{ab}(r))\theta_b(r)$

$M_{ab}(r) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \tilde{M}_{ab}(\vec{k})$

Sigma model (gradient expansion)



$\psi_a(r) = g_{ab}(r)\theta_b(r)$, $g_{ab}(r)$ unitary

$\mathcal{S}$ local functional of derivatives of g_{ab}.

$$\mathcal{S} = g \int dt \int d^{(2)}r \ \mathrm{Tr} \left[(\partial_t g)^2 - (\partial_x g)^2 - (\partial_y g)^2 \right]$$

An $U(d)$ σ-model for collective dynamics ? (II)

Projection on a space of holomorphic functions not compatible with unitarity condition $\sum_b g_{ba}(r)\overline{g_{bc}(r)} = \delta_{ac}$.
Our “spin-wave theory” has the following structure:
$\psi_a(r) = \left[(\delta_{ab} + \hat{M}_{ab})\theta_b\right](r)$ with $\hat{M}_{ab}(r) = \mathcal{P}_{hol}\left(\sum_{\vec{k}} M_{ab}(\vec{k})e^{i\vec{k}\cdot\vec{r}}\right)$

Suggests to construct gradient expansion using $\mathcal{P}_{hol}$:
$\psi_a(r) = \mathcal{P}_{hol}\left(g_{ab}(r)\theta_b\right)(r)$?
Note: $\mathcal{P}_{hol}f\mathcal{P}_{hol}g\theta = \mathcal{P}_{hol}(f\star g)\theta$

But is there an optimal choice of $\mathcal{P}_{hol}$?
$\mathcal{S}$ non-local functional of derivatives of g_{ab}. Can we approximate it by a local one in the long wave-length limit ?

Summary (I)

- Construction of Slater determinants in L.L.L associated to smooth classical spin textures.
- Use of a semi-classical expansion in the $l \to 0$ limit.
- Heuristic picture: Slater determinants associated to smooth spin textures as coherent states in fermionic Fock space.
- $\mathbb{C}P(d-1)$ model emerges as principal symbol of low-energy Hamiltonian H_{eff}.
- Highly degenerate ground-state spanned by holomorphic textures.
- Degeneracy robust to the introduction of quantum fluctuations.

Summary (II)

- The anti-holomorphic degrees of freedom have a finite but small energy gap, of order nl^2.
- Degeneracy among holomorphic textures is lifted by long-range tail of interaction potential (sub-principal symbol of H_{eff}).
- Yields Skyrmion crystals which spontaneously break $SU(d)$ symmetry.
- Existence of collective (Goldstone) modes similar to those in non-collinear antiferromagnets.

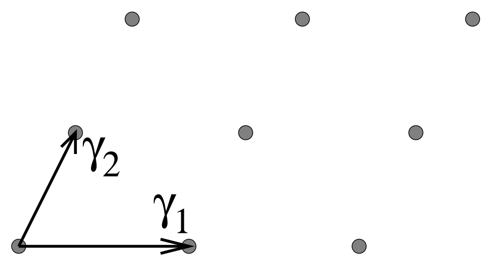
Open questions

- Small Hessian gap $\mathcal{O}(nl^2)$ associated to anti-holomorphic modes $\longrightarrow$ can we justify projection onto the linear span of holomorphic textures, when the sub-principal symbol of $H_{\rm eff}$ is introduced ?
- Are the collective degrees of freedom described by an emerging $U(d)$ σ-model ?
- Role of non-commutativity of physical plane ?
- Role of quantum fluctuations $\longrightarrow$ quantum melting of Skyrmion crystal?
- Connection to experiments (NMR relaxations in bilayers)?
- Extension to higher integer filling factors $\longrightarrow$ $\mathbb{C}P^{(d-1)}$ replaced by Grassmanian manifolds.

Construction of periodic textures

Problem: construct periodic holomorphic maps from torus to projective space
Answer: use Theta functions

$$
\begin{aligned}
\gamma_1 &= \pi\sqrt{d} \\
\gamma_2 &= \pi\sqrt{d}\tau
\end{aligned}
$$



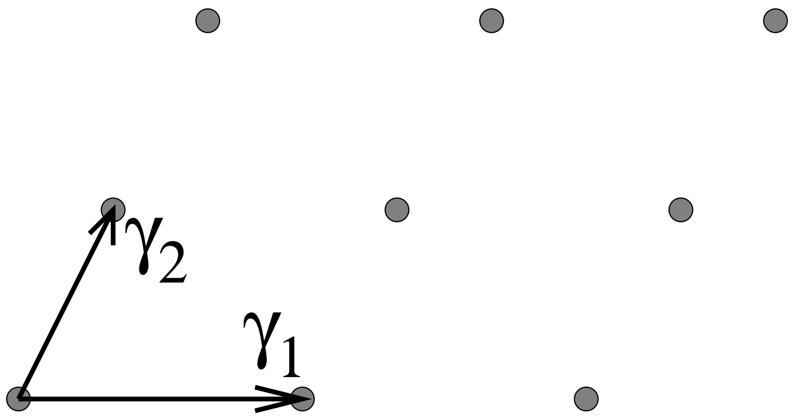
$$\theta(z+\gamma) = e^{a_\gamma z + b_\gamma}\theta(z)$$

$$\gamma = n_1\gamma_1 + n_2\gamma_2$$

n_1 and n_2 integers

Fixing the topological charge d

$$\frac{1}{i}\int_{\mathcal{C}(\gamma_1,\gamma_2)}\frac{\theta'(z)}{\theta(z)} = \frac{1}{i}\left(a_{\gamma_1}\gamma_2 - a_{\gamma_2}\gamma_1\right) = 2\pi d$$



Theta functions of a fixed type carrying topological charge d on the elementary (γ_1, γ_2) parallelogram form a complex vector space of dimension d (Riemann Roch theorem on torus).

Lattice of allowed translations

$$\mathcal{T}_w\theta(z) = e^{\mu(w)z}\theta(z-w)$$

$$\frac{\mathcal{T}_w\theta(z+\gamma)}{\mathcal{T}_w\theta(z)} = e^{a_\gamma z+b_\gamma}e^{\mu(w)\gamma-a_\gamma w}$$

Type conservation:

$$\mu(w)\gamma - a_\gamma w \in 2\pi\mathbb{Z}$$

for any lattice vector γ.

Quantized translations:

$$\begin{aligned} w &= \frac{1}{d}(m_1\gamma_1 + m_2\gamma_2) \\ \mu(w) &= \frac{1}{d}(m_1 a_{\gamma_1} + m_2 a_{\gamma_2}) \end{aligned}$$

$$\mathcal{T}_w\mathcal{T}_{w'} = e^{i\frac{2\pi}{d}(m_1m_2'-m_2m_1')}\mathcal{T}_{w'}\mathcal{T}_w$$

$(m_1m_2' - m_2m_1')/d =$ topological charge inside parallelogram delimited by w and w'.

Useful set of theta functions

$$\theta_p(z) = \sum_n e^{i\left(\pi\tau d(n-p/d)(n-1-p/d)+2\sqrt{d}(n-p/d)z\right)}$$

Pattern of zeros (d=4)

$$\begin{aligned}\mathcal{T}_{\frac{\gamma_1}{d}}\theta_p &= e^{i\frac{2\pi p}{d}}\theta_p \\ \mathcal{T}_{\frac{\gamma_2}{d}}\theta_p &= \lambda\theta_{p+1}\end{aligned}$$

$$\lambda = \exp\left(-i\pi\tau(d+1/d)\right)$$

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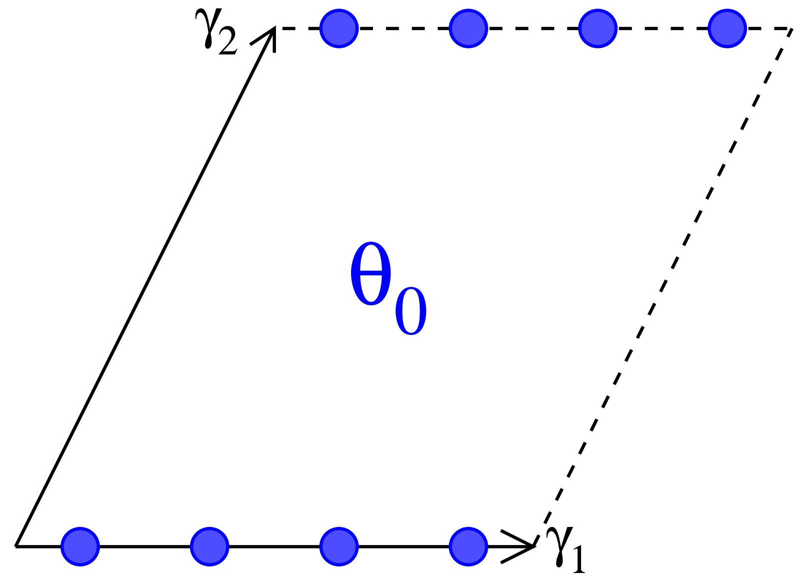

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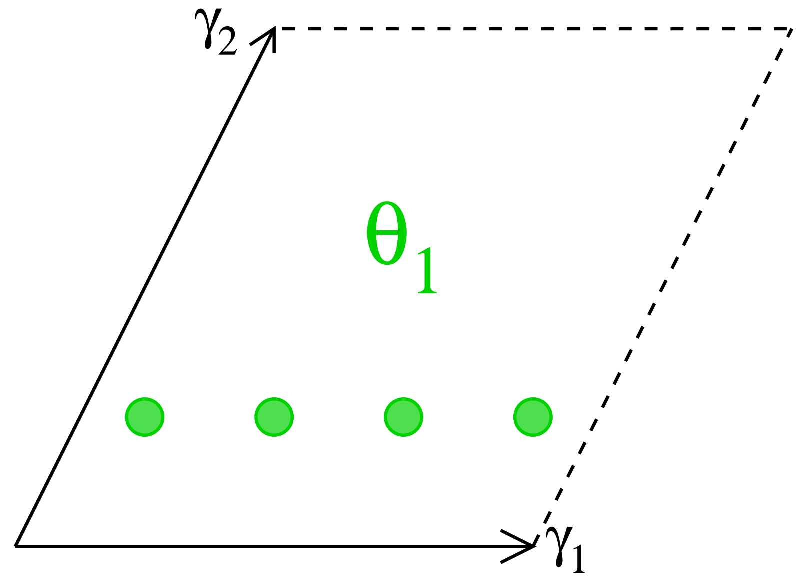

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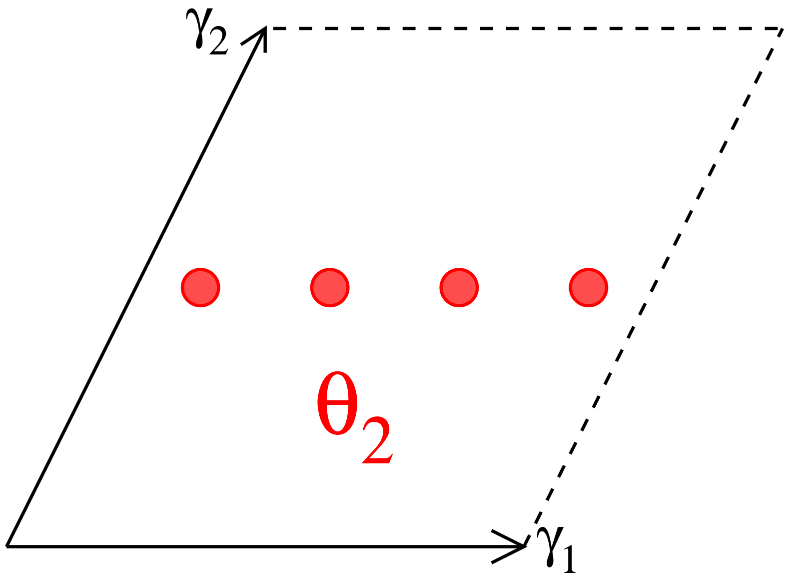

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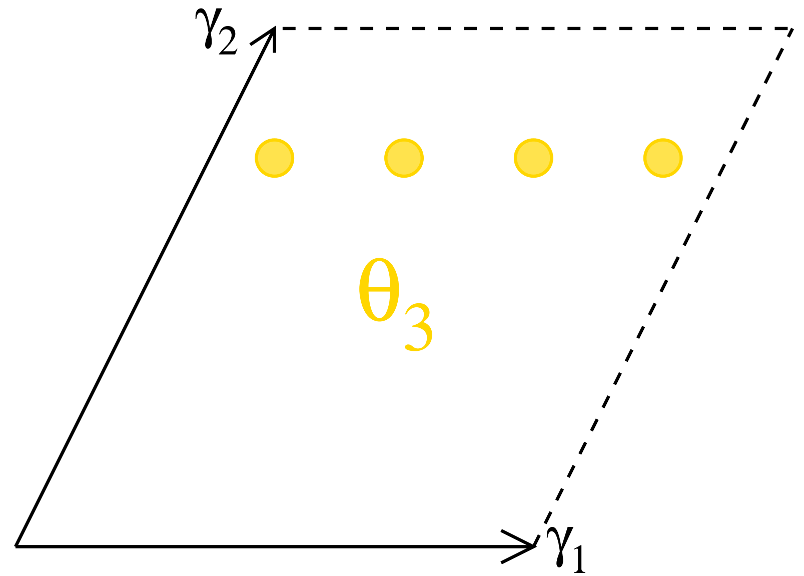
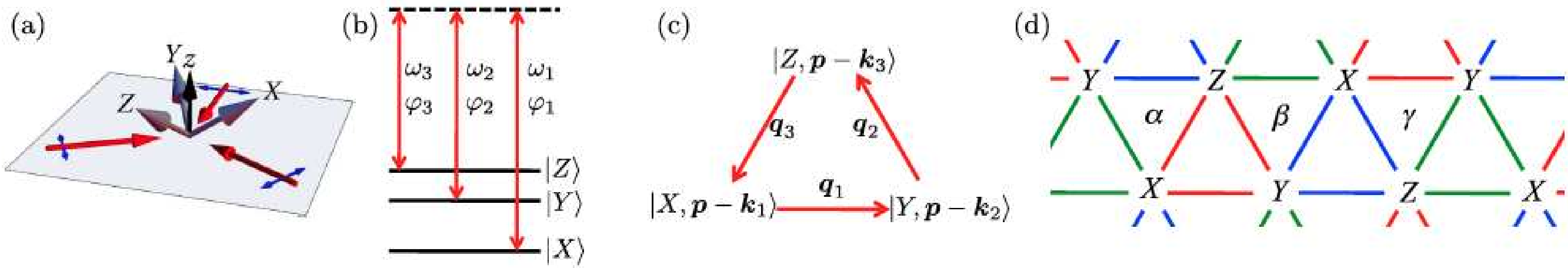


Applications of a flat topological charge profile



N. Cooper and J. Dalibard, PRL **110**, 185301 (2013); N. Cooper and R. Moessner, PRL **109**, 215302 (2012)

Tight binding model in momentum space with a non-zero average flux (*à la* Hofstadter) corresponds, in the large N limit to a periodic texture in real space $r \rightarrow |\psi(r)\rangle$ with very flat Berry curvature. After adding kinetic energy of atoms, this generates a very flat effective orbital magnetic field.

For $N = 3$, $\Omega = 3E_{\mathrm{R}}$, get Landau level with a bandwidth

$W = 0.015E_{\mathrm{R}}$.