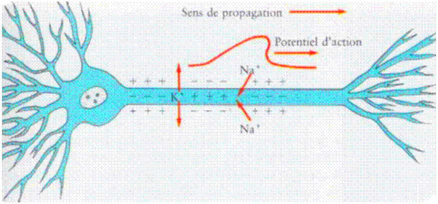
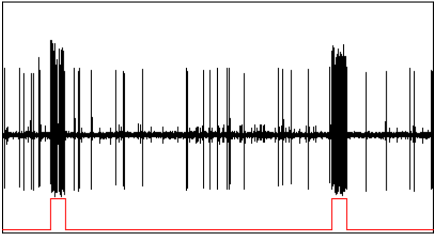
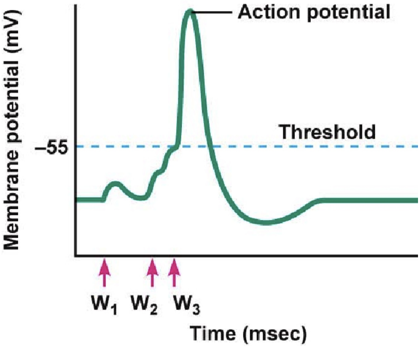
Estimation of the spiking rate for interacting neurons

P. Hodara, N. Krell and E. Löcherbach

Membrane potential of a neuron

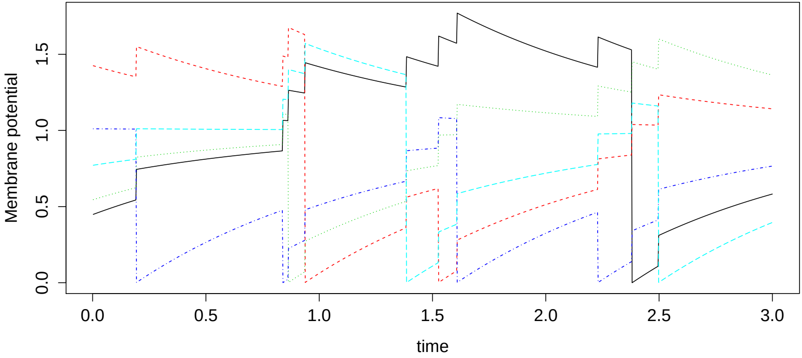






The dynamics

fig. 1



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$$\begin{aligned} X_t^i &= X_0^i - \lambda \int_0^t (X_s^i - m) ds - \int_0^t \int_0^\infty X_{s-}^i 1_{\{z \le f(X_{s-}^i)\}} N^i(ds, dz) \\ &+ \frac{1}{N} \sum_{j \neq i} \int_0^t \int_0^\infty 1_{\{z \le f(X_{s-}^j)\}} N^j(ds, dz). \end{aligned}$$

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$$L\varphi(x) = \sum_{i=1}^N f(x_i) \left[\varphi(\Delta_i(x)) - \varphi(x)\right] - \lambda \sum_i \left(\frac{\partial \varphi}{\partial x_i}(x) \left[x_i - m\right] \right),$$

with

$$(\Delta_i(x))_j = \left\{ \begin{array}{ll} x_j + \frac{1}{N} & j \neq i \\ 0 & j = i \end{array} \right\}.$$

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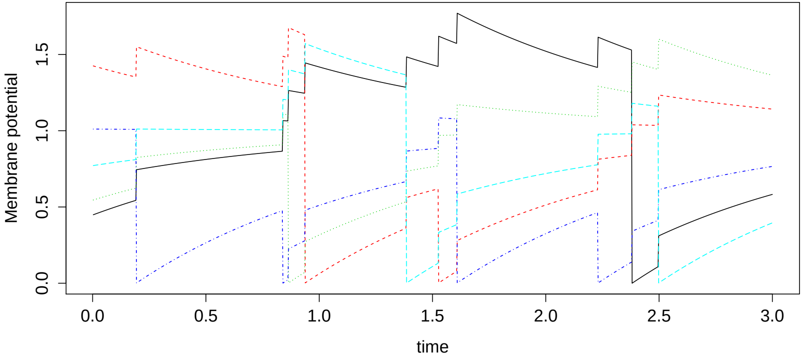

Construction of the estimator

Define the jump times :

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jump measure : $\mu(dt, dx) = \sum_{i=1}^{N} \mu^i(ds, dx).$

occupation time measure : $\eta(A \times B) = \int_A \left(\sum_{i=1}^{N} 1_B(X_s^i) \right) ds$

Construction of the estimator

We define the estimator using a Kernel Q with compact support satisfying

(1) $$Q \in C_c(\mathbb{R}_+), \int_{\mathbb{R}_+} Q(y)dy = 1,$$

in the following way :

(2)

$$\hat{f}_{t,h}(a) = \frac{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\mu(ds,dy)}{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\eta(ds,dy)}, \text{ with } Q_h(y) := \frac{1}{h} Q\left(\frac{y}{h}\right).$$

Assumptions

- $f(0) = 0$, f is non-decreasing and belongs to the following Hölderclass of order $\beta = k + \alpha$:

$$H(\beta, F, L, f_{min}) = \{ f \in C^k(\mathbb{R}_+) : \\ f(x) \geq f_{min}(x) \text{ for allt } x, \\ |\frac{d^l}{dx^l} f(x)| \leq F, \text{ for all } 1 \leq l \leq k, x \in \mathbb{R}_+, \\ |f^{(k)}(x) - f^{(k)}(y)| \leq L|x-y|^\alpha \text{ for all } x, y \}.$$

- $\int_{\mathbb{R}_+} Q(y) y^j dy = 0$ for all $1 \leq j \leq k$, and $\int_{\mathbb{R}_+} |y|^\beta Q(y) dy < \infty$,

Ergodicity

Theorem

The process X is positif Harris recurrent with unique invariant measure π, i.e. for all $B \in \mathcal{B}([0, K]^N)$,

$$\pi(B) > 0 \text{ implies } P_x \left(\int_0^\infty 1_B(X_s) ds = \infty \right) = 1$$

for all $x \in [0, K]^N$.
Moreover, there exists constants $C > 0$ and $\kappa > 1$ such that

$$\sup_{f \in H(\beta, F, L, f_{min})} \| P_t(x, \cdot) - \pi \|_{TV} \leq C \kappa^{-t}.$$

Regularity of the invariant measure

For $d > \frac{(k+2)}{N}$, we define :

$$S_{d,k} := \{w \in \mathbb{R}_+ : \frac{k}{N} < w < K - \frac{k}{N}, |w - m| > d\},$$

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Theorem (Löcherbach (2016))

Let us define $\pi^1 := \mathcal{L}_\pi(X_t^1)$, i.e. $\int g d\pi^1 = E_\pi(g(X_t^1))$, then π^1 has a bounded probability density function π_1 with respect to the Lebesgue measure on $S_{d,k}$. Moreover, $\pi_1 \in C^k(S_{d,k})$ and

$$\sup_{\ell \leq k, w \in S_{d,k}} |\pi_1^{(\ell)}(w)| + \sup_{w \neq w', w, w' \in S_{d,k}} \frac{\pi_1^{(k)}(w) - \pi_1^{(k)}(w')}{|w - w'|^\alpha} \leq C_F,$$

where the constant C_F depends on d and on the class $H(\beta, F, L, f_{min})$, but of nothing else.

Definitions

Recall the definition of the estimator :

$$\hat{f}_{t,h}(a) = \frac{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\mu(ds,dy)}{\int_0^t \int_{\mathbb{R}} Q_h(y-a)\eta(ds,dy)}.$$

This estimator is well defined on events of the type

$$A_{t,r} := \left\{ \frac{1}{Nt} \int_0^t \int_{\mathbb{R}} Q_h(y-a)\eta(ds,dy) \geq r \right\}.$$

Convergence of the estimator

Theorem

There exists $r^* > 0$ such that for all $a \in S_{d,k}$ we have,

(i) for $h_t := t^{-\frac{1}{2\beta+1}}$, for all $x \in [0, K]^N$,

$$\limsup_{t \to \infty} \sup_{f \in H(\beta, F, L, f_{min})} t^{\frac{2\beta}{2\beta+1}} E_x^f \left[|\hat{f}_{t,h_t}(a) - f(a)|^2 | A_{t,r^*} \right] < \infty.$$

(ii) Moreover, for $h_t = o(t^{-1/(1+2\beta)})$, for all $f \in H(\beta, F, L, f_{min})$ and $a \in S_{d,k}$, we have the following weak convergence under P_x^f :

$$\sqrt{th_t}\left(\hat{f}_{t,h_t}(a) - f(a)\right) \to \mathcal{N}(0, \Sigma(a))$$

with $\Sigma(a) = \frac{f(a)}{N\pi_1(a)} \int Q^2(y)dy$.

Optimal speed of convergence

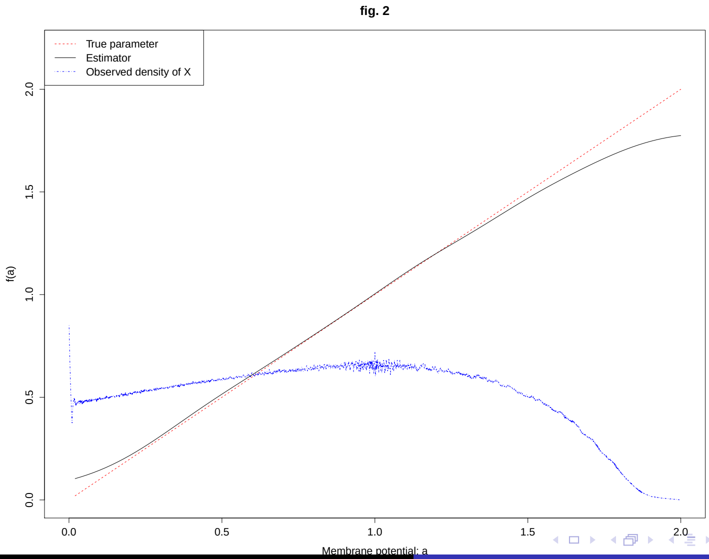
Theorem

For all $a \in S_{d,k}$ and $x \in [0, K]^N$, we have

$$\lim_{t\to\infty} \inf \inf_{\hat{f}_t} \sup_{f \in H(\beta, F, L, f_{min})} t^{\frac{2\beta}{1+2\beta}} E_x^f [|\hat{f}_t(a) - f(a)|^2] > 0,$$

where the inf is considered on the set of all possible estimators $\hat{f}_t(a)$ for $f(a)$.

Practical results on simulations



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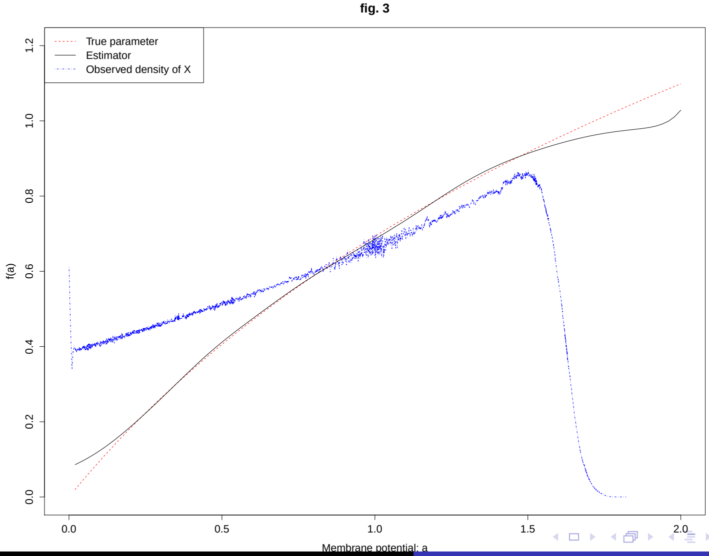

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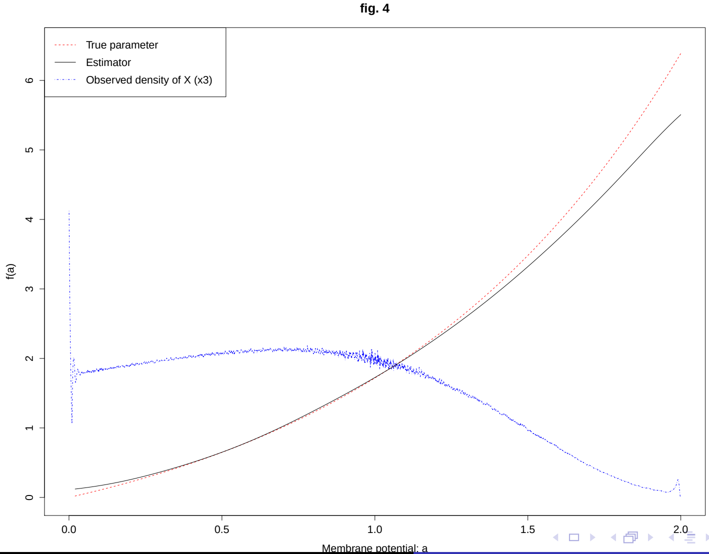

$$\left(\frac{1}{Nt}\int_{\mathbb{R}}\frac{1}{h}Q\left(\frac{y-a}{h}\right)\eta_t(dy)\right)\left(\hat{f}_{t,h}(a)-f(a)\right)$$

$$(3)\quad = \frac{1}{Nt}\int_{\mathbb{R}} Q_h(y-a)\mu_t(dy)f(y) - \frac{1}{Nt}\int_{\mathbb{R}} Q_h(y-a)f(y)\eta_t(dy)$$

$$(4)\quad + \frac{1}{Nt}\int_{\mathbb{R}} Q_h(y-a)\Big(f(y)-f(a)\Big)\eta_t(dy) - \int_{\mathbb{R}} Q_h(y-a)\Big(f(y)-f(a)\Big)\pi_1(dy)$$

$$(5)\quad + \int_{\mathbb{R}} Q_h(y-a)\Big(f(y)-f(a)\Big)\pi_1(dy).$$

Thank you !