Smoothness of the invariant density of interacting neurons

Eva Löcherbach

Rennes, Dynstoch 2016

Interacting neurons

- N neurons $X_t^1, \ldots, X_t^N, X_t^i \in \mathbb{R}, t \geq 0$
- Each neuron ‘spikes’ with rate $f(X_t^i)$.
- $f \in C^1$, strictly positive.

Interacting neurons

- N neurons $X_t^1, \ldots, X_t^N, X_t^i \in \mathbb{R}, t \geq 0$
- Each neuron 'spikes' with rate $f(X_t^i)$.
- $f \in C^1$, strictly positive.
- If i spikes :

$\Rightarrow$ neuron i is reset to a resting potential 0
$\Rightarrow$ for all $j \neq i$: j receives an additional amount of potential $W_{i \to j}$.

- Between two successive spikes, some leak effect induces an attraction to an equilibrium potential m

$$dX_t^i = b(X_t^i)dt = -\lambda(X_t^i - m)dt, \lambda > 0.$$

• Between two successive spikes, some leak effect induces an attraction to an equilibrium potential m

$$dX_t^i = b(X_t^i)dt = -\lambda(X_t^i - m)dt, \lambda > 0.$$

Remark

Process is a PDMP with generator

$$Lg(x) = \sum_{i=1}^N f(x^i)[g(\Delta_i(x)) - g(x)] - \lambda \sum_{i=1}^N \frac{\partial g}{\partial x^i}(x^i - m),$$

where

$$\Delta_i(x) = \left(x^1 + W_{i\to 1}, ..., x^{i-1} + W_{i\to i-1}, 0, x^{i+1} + W_{i\to i+1}, ...\right)^T .$$

Where does this model come from ?

• Can be seen as a very easy variant of Leaky Integrate and Fire Models where spiking occurs randomly with a rate depending on the potential.

• There is some relation with interacting Hawkes processes having memory of variable length...

AIM

- Want to estimate $f(a)$, based on observation of $(X_t)_{t\in[0,T]}$, in a non-parametric way.

- Next talk : Kernel type estimator.

- Need : Regularity of invariant density !

Hypothesis

X recurrent in the sense of Harris, with invariant probability measure π.

(Follows basically from **partial regeneration** induced by spikes.)

- $\pi^1(g) := \int_{\mathbb{R}^N} \pi(dx) g(x^1)$ invariant probability of first neuron.
- Question : $\pi(dx) = \pi(x)dx$? π smooth ? ? ? ?
- Question : $\pi^1(dx^1) = \pi^1(x^1)dx^1$? π^1 smooth ? ? ? ?

- $\pi^1(g) := \int_{\mathbb{R}^N} \pi(dx) g(x^1)$ invariant probability of first neuron.
- Question : $\pi(dx) = \pi(x)dx$? π smooth ? ? ? ?
- Question : $\pi^1(dx^1) = \pi^1(x^1)dx^1$? π^1 smooth ? ? ? ?

Problem : There is **not** a lot of noise in the system. Only the "exponential densities" of the jump times.

Second Problem : Jump kernel $K(x, dy) = \sum_{i=1}^N \frac{f(x^i)}{\bar{f}(x)} \delta_{\Delta^i(x)}(dy)$, $\bar{f} = \sum_{j=1}^N f(x^j)$, is partly degenerate ! Indeed : $[\Delta_i(x)]^i = 0$!!!!

IPP based on jump noise

Proposition

Let $\gamma_t(x)$ the joint flow of the N particles starting from $x \in \mathbb{R}^N$ at time 0 (solution to ODE), $e(t,x) = e^{-\int_0^t \bar{f}(\gamma_s(x))ds}$ survival rate. Then

$$\pi(g) = \sum_{i=1}^N \int_{\mathbb{R}^N} \pi(dx) f(x^i) \int_0^\infty e(t, \Delta_i(x)) g(\gamma_t(\Delta^i(x)) dt.$$

(Follows from considering the "just-before-jump" chain and its transitions)

Application

This implies for the first particle :

$$E_\pi(h'(X_t^1)) = \sum_{i=1}^N \int_{\mathbb{R}^N} \pi(dx) f(x^i) \int_0^\infty e(t, \Delta_i(x)) h'(\gamma_t^1(\Delta^i(x)))dt.$$

But $(y = [\Delta^i(x)]^1)$

$$\int_0^\infty e(t,y) h'(\gamma_t^1(y)dt = \int_0^\infty \frac{e(t,y)}{b(\gamma_t^1(y))} [h \circ \gamma_t^1]'(y)dt$$
$$= \left[e(t,y) \frac{h(\gamma_t^1(y))}{b(\gamma_t^1(y))} \right]_{t=0}^{t=\infty} - \int_0^\infty \frac{d}{dt} \left(\frac{e(t,y)}{b(\gamma_t^1(y))} \right) h(\gamma_t^1(y))dt.$$

1. PROBLEM : border term at $t = 0$ gives $\frac{h(y)}{b(y)}$ where $y = [\Delta^i(x)]^1$ position of first neuron after a spike of i. If $i = 1$ this gives the total contribution

$$\int \pi(dx) f(x^1) \frac{h(0)}{b(0)} : \text{ Dirac measure in 0!}$$

$\Longrightarrow$ have to stay away from 0 ! ! !

1. PROBLEM : border term at $t = 0$ gives $\frac{h(y)}{b(y)}$ where $y = [\Delta^i(x)]^1$ position of first neuron after a spike of i. If $i = 1$ this gives the total contribution

$$\int \pi(dx) f(x^1) \frac{h(0)}{b(0)} : \text{ Dirac measure in 0!}$$

$\Longrightarrow$ have to stay away from 0 ! ! !

2. PROBLEM : we divide by $b(\gamma_t^1(y))$.
$\Longrightarrow$ have to stay away from $\{y : b(y) = 0\} = \{m\}$.

Theorem

Let $f \in C^k, \|f\|_{\infty,k} \leq F$, such that $f^{(k)}$ is Hölder α. Then

$$\pi^1 \in C^k(\Omega_k)$$

and

$$\sup_{v \neq v', v, v' \in \Omega_k} \frac{|(\pi^1)^{(k)}(v) - (\pi^1)^{(k)}(v')|}{|v - v'|^\alpha} \leq C,$$

where C does not depend on f but only on the bounds of the function class f belongs to.

Here, Ω_k denotes the subset of all positions "sufficiently far away" from 0 and from m, **even after k IPP's!**

Outlook : Lebesgue density in dimension N

First comments :

- Since we can only use the jump noise, we need at least N jumps.
- The flow transports (preserves) density nicely.
- Jump of particle i destroys density in direction of e_i.

Outlook : Lebesgue density in dimension N

First comments :

- Since we can only use the jump noise, we need at least N jumps.

- The flow transports (preserves) density nicely.

- Jump of particle i destroys density in direction of e_i. But :
Immediately after, density is created by the jump noise.

NUMMELIN SPLITTING

For the “just-before-jump”-chain $Z_k = X_{T_k -}$, with associated transition kernel Q :

Theorem

$$Q^N(x, dy) \geq 1_C(x)\beta\nu(y)dy,$$

where $\nu \in C_c^\infty(\mathbb{R}^N)$.

• Idea of proof : A specific **order of successive jumps** creates density : e.g. 1 spikes first, followed by 2 followed by 3 etc...

• Idea of proof : A specific **order of successive jumps** creates density : e.g. 1 spikes first, followed by 2 followed by 3 etc... this could be compared to the weak Hörmander condition. Successive jumps of $1, 2, 3, \ldots$ induce a diagonal structure of what would be the "Malliavin covariance matrix" here.

• The idea of using favorable sequences of jump events has already been used by Duarte and Ost (2015) to show Harris recurrence of the process.

• Nummelin splitting implies : there exists an extended stopping time (the regeneration time) R such that

$$X_{T_R-} = Z_R \sim \nu(x)dx.$$

• Can we preserve this density ? ? ?

Preservation of density

We start from

$$X_{T_R-} = Z_R \sim \nu(x)dx.$$

Suppose i jumps at time $T_R \Rightarrow$ replace $x \mapsto \Delta_i(x)$: does not depend on x^i any more. But :

$$(t, x) \mapsto G(t, x) = \gamma_t(\Delta_i(x)) \text{ explores the whole space :}$$

$$J_G(t, x) = \det \sqrt{\frac{\partial G}{\partial t \partial x} \left(\frac{\partial G}{\partial t \partial x} \right)^T} > 0.$$

Preservation of density

We start from

$$X_{T_R -} = Z_R \sim \nu(x)dx.$$

Suppose i jumps at time $T_R \Rightarrow$ replace $x \mapsto \Delta_i(x)$: does not depend on x^i any more. But :

$$(t, x) \mapsto G(t, x) = \gamma_t(\Delta_i(x)) \text{ explores the whole space :}$$

$$J_G(t, x) = \det \sqrt{\frac{\partial G}{\partial t \partial x} \left(\frac{\partial G}{\partial t \partial x}\right)^T} > 0.$$

In this case, the **co-area formula** implies that we have a measurable Lebesgue density for Z_{R+1} and thus for Z_n for all $n \geq R$. In particular, $\pi(dx) = \pi(x)dx$ with some **measurable** π.

In order to obtain more regularity, we have to work more (no IPP, but transformations of variables - based on the flow for the non-spiking particles, and based on the jump noise for the spiking one) :

Theorem

If $f \geq f_0 > \lambda$, the invariant density π is at least $k-$times differentiable, for any $k : 2k < Nf_0/\lambda - N$.

So we need a balance between the explosion rate λ of the inverse flow and the minimal jump rate.

This is of course a very strong condition - but the transitions are also very degenerate...

Some literature

• DUARTE, A., OST, G. A model for neural activity in the absence of external stimuli. To appear in Markov Proc. Related Fields 2016, available on http ://arxiv.org/abs/1410.6086.

• POLY, G. Absolute continuity of Markov chains ergodic measures by Dirichlet forms methods. To appear in Ann. IHP, 2013.

• You can find this work on arXiv :
https ://arxiv.org/abs/1601.07123, 2016.

Thank you for your attention.