Jump filtering and efficient drift estimation for Lévy-driven SDE's

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Outline

Outline

Model I

Let $\Theta \subset \mathbb{R}^d$, Θ-compact. We aim at estimating the unknown drift parameter $\theta \in \Theta$ of a jump diffusion process X^θ given by

$$X_t^\theta = X_0^\theta + \int_0^t b(\theta, X_s^\theta)\, ds + \int_0^t \sigma(X_s^\theta)\, dW_s + \int_0^t \gamma(X_{s-}^\theta)\, dL_s$$

where $t \in \mathbb{R}_+$, $W = (W_t)_{t\geq 0}$ is a one-dimensional Brownian motion and L a pure jump Lévy process with Lévy measure ν, such that

$$\int_{\{0<|z|\leq 1\}} |z|\nu(dz) < \infty.$$

Sampling scheme

High frequency data with an observation time going to infinity:

$$0 \leq t_0 \leq \ldots \leq t_n \qquad X^{\theta}_{t_0}, \ldots, X^{\theta}_{t_n}$$

such that

$$\Delta_n := max\{t_i - t_{i-1} : 1 \leq i \leq n\} \to 0, \qquad \text{as} \quad n \to \infty;$$

$$t_n \to \infty \qquad \text{and} \qquad t_n = O(n\Delta_n).$$

Goals:

- efficient estimation of the drift parameter,
- minimal conditions on the sampling step Δ_n.

Literature about the high frequency inference for diffusion with jumps

- [Masuda (13)]: Gaussian quasi-likelihood estimators
- [Shimizu and Yoshida (06)]: contrast-type estimation function, jumps of compound Poisson type.
- [Shimizu (06)]: include more general driving Lévy processes.
- [Tran(14)]: LAN property for drift and diffusion parameters via Malliavin calculus.
- [Mai(2014)]: drift estimation for Lévy-driven Ornstein-Uhlenbeck.

except [Mai(2014)], joint estimation of the drift, diffusion and jump part parameters is considered;
under condition which is at best

$$n\Delta_n^2 \to 0.$$

- The estimation of the volatility is feasible on a compact interval,
- the estimation of the drift and the jump law requires a growing time window.
- Due to the Poisson structure of the jump part the estimation of the jump parameter can be well separated from those of the drift and the diffusion.

We focus on the estimation of the drift parameter only; and construct a consistent, asymptotically normal and efficient estimator, under conditions

$$n\Delta_n^{3-\varepsilon} \to 0.$$

Remark: the condition $n\Delta_n^3 \to 0$ was found by [Florens-Zimrou(89)] and [Yoshida(92)] in the case of drift estimation for continuous diffusions.

Outline

The equation of the model can be rewritten as

$$X_t^\theta = X_0^\theta + \int_0^t b(\theta, X_s^\theta)\, ds + \int_0^t \sigma(X_s^\theta)\, dW_s + \int_0^t \int_{\mathbb{R}} \gamma(X_{s-}^\theta) z \mu(ds, dz)$$

where μ is the Poisson random measure on $[0, \infty) \times \mathbb{R}$,

$$L_t = \int_0^t \int_{\mathbb{R}} z \mu(ds, dz)$$

is the Lévy process with Lévy-Khintchine triplet $(0, 0, \nu)$ such that

$$\int_{\{0<|z|\leq 1\}} |z| d\nu(z) < \infty.$$

X_0^θ, W and L are independent.

Assumption (Existence)

Assumption (Irreducibility)

Assumption (Non-degeneracy)

There exists some $\alpha > 0$, *such that* $\sigma^2(x) \geq \alpha$ *for all* $x \in \mathbb{R}$.

Assumption (Identifiability)

Assumption (Hölder-continuity of the drift and its 1,2 derivatives with respect to θ.)

Assumption (Subpolynomial growth of all Hölder constants)

Assumption (Jumps)

The jump coefficient γ *is bounded from below;*
If $\nu(\mathbb{R}) = \infty$, $\int_{0<|z|\leq 1} |z|\nu(dz) < \infty$, *the Lévy measure* ν *is absolutely continuous with respect to the Lebesgue measure, and* γ *is upper bounded.*

Assumption (Ergodicity)

(i) *For all* $q > 0$, $\int_{|z|>1} |z|^q \nu(dz) < \infty$.

(ii) *For all* $\theta \in \Theta$ *there exists a constant* $C > 0$ *such that* $xb(\theta, x) \leq -C|x|^2$, *if* $|x| \to \infty$.

(iii) $|\gamma(x)|/|x| \to 0$ *as* $|x| \to \infty$.

(iv) $|\sigma(x)|/|x| \to 0$ *as* $|x| \to \infty$.

(v) $\forall \theta \in \Theta$, $\forall q > 0$ *we have* $\mathbb{E}|X_0^\theta|^q < \infty$.

The last Assumption ensure the existence of unique invariant distribution π^θ, as well as the ergodicity of the process X^θ, similarly to [Masuda(2007)].

Lemme

For all $\theta \in \Theta$, X^θ admits a unique invariant distribution π^θ and the ergodic theorem holds:

1. *for every measurable function $g : \mathbb{R} \to \mathbb{R}$ satisfying $\pi^\theta(g) < \infty$, we have $a.s.$*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(X_s^\theta) ds = \pi^\theta(g).$$

2. *For all $q > 0$, $\pi^\theta(|x|^q) < \infty$.*
3. *For all $q > 0$, $\sup_{t\in\mathbb{R}} E[|X_t^\theta|^q] < \infty$.*
4. *Moreover,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t E[|X_s^\theta|^q] ds = \pi^\theta(|x|^q).$$

Outline

Construction of the estimator

The likelihood function is given by

$$\mathcal{L}_t(\theta, X) = \frac{dP_t^\theta}{dP_t^0}(X) =$$

$$\exp\left(\int_0^t \sigma(X_s)^{-2} b(\theta, X_s)\, dX_s^c - \frac{1}{2}\int_0^t \sigma(X_s)^{-2} b(\theta, X_s)^2\, ds\right).$$

We define the log-likelihood function as

$$\ell_t(\theta) := \ln \mathcal{L}_t(\theta, X).$$

The problem is that X^c is unobserved !

Our aim is to approximate $\ell_t(\theta)$ from discrete sample and thus define some contrast.

Define the increment's operator Δ_i^n :

$$\Delta_i^n X = X_{t_i} - X_{t_{i-1}}, \quad \Delta_i^n X^c = X_{t_i}^c - X_{t_{i-1}}^c \quad \Delta_i^n Id = t_i - t_{i-1}.$$

Let $\varepsilon \in (0, 1/2)$ and denote

$$v_n = \Delta_n^{1/2-\varepsilon},\ n \geq 1.$$

$$\ell_{t_n}^n(\theta) = \sum_{i=1}^n \frac{b(\theta, X_{t_{i-1}})}{\sigma(X_{t_{i-1}})^2} \Delta_i^n X \mathbf{1}_{\{|\Delta_i^n X| \leq v_n\}} - \frac{1}{2} \sum_{i=1}^n \frac{b(\theta, X_{t_{i-1}})^2}{\sigma(X_{t_{i-1}})^2} \Delta_i^n Id.$$

Finally, we define an estimator $\hat{\theta}_n$ of the true value $\theta^\star$ as

$$\hat{\theta}_n \in \underset{\theta \in \Theta}{\operatorname{argmax}}\, \ell_{t_n}^n(\theta)$$

and in the sequel we call it the filtered MLE (FMLE).

Outline

Main results

without further assumptions on n, Δ_n and v_n.

Théorème (Consistency)

The FMLE $\hat{\theta}_n$ is consistent in probability:

$$\hat{\theta}_n \xrightarrow{P} \theta^\star, \qquad n \to \infty.$$

Define the asymptotic Fisher information by

$$I(\theta) = \left(\int_{\mathbb{R}} \frac{\partial_{\theta_i} b(\theta, x) \partial_{\theta_j} b(\theta, x)}{\sigma^2(x)} \pi^\theta(dx) \right)_{1 \leq i,j \leq d}. \qquad (1)$$

Assumption

For all $\theta \in \Theta$, $I(\theta)$ is non-degenerated.

Théorème (Asymptotic normality: finite activity :$\nu(\mathbb{R}) < \infty$)

If $n\Delta_n^{3-\varepsilon} \to 0$, $\sqrt{n}\Delta_n^{1-\varepsilon/2}\left(\int_{|z|\le 2v_n} \nu(dz)\right)^{1-\varepsilon/2} \to 0$ *and* $\sqrt{n}\Delta_n^{1/2}\int_{|z|<2v_n} |z|\nu(dz) \to 0$ *as* $n \to \infty$, *then*

$$t_n^{1/2}(\hat{\theta}_n - \theta^\star) \xrightarrow{\mathcal{L}} N(0, I^{-1}(\theta^\star)), \quad n \to \infty,$$

Furthermore, the FMLE $\hat{\theta}_n$ *is asymptotically efficient in the sense of the Hàjek-Le Cam convolution theorem.*

Remarque

If ν has a bounded Lebesgue density, all the conditions reduce to $n\Delta_n^{3-4\varepsilon} \to 0$.

Théorème (Asymptotic normality: general case: $\nu(\mathbb{R}) <= \infty$)

If $n\Delta_n^{3-\varepsilon} \to 0$,

$$\sqrt{n\Delta_n}\left(\int_{|z|\leq 3v_n/\gamma_{min}} |z|\nu(dz)\right)^{1-\varepsilon/2} \to 0$$

and

$$\sqrt{n}\Delta_n^{3/2-2\varepsilon}\left(\int_{|z|\geq 3v_n/\gamma_{min}} \nu(dz)\right)^{1-\varepsilon/2} \to 0$$

as $n \to \infty$, *then*

$$t_n^{1/2}(\hat{\theta}_n - \theta^\star) \xrightarrow{\mathcal{L}} N(0, I^{-1}(\theta^\star)), \quad n \to \infty,$$

Furthermore, the FMLE $\hat{\theta}_n$ *is asymptotically efficient in the sense of the Hàjek-Le Cam convolution theorem.*

In the case where ν admits a bounded Lebesgue density, all the conditions on the Δ_n and n of the Theorem became $n\Delta_n^{3-\tilde{\varepsilon}} \to 0$ for some $\tilde{\varepsilon} > 0$.

Example (tempered stable jumps)

The Lévy measure in this case has an unbounded and non-integrable density given by

$$\nu(dz) = C|z|^{-(1+\alpha)}e^{-\lambda|z|}dz$$

with $\lambda > 0$ and $C > 0$ satisfies the conditions of the previous Theorem if $0 < \alpha < 1$.
The conditions on n, Δ_n and ν can now be summarized as $n\Delta_n^{2-\alpha-\tilde{\epsilon}} \to 0$ for some $\epsilon > 0$. We observe that a higher Blumenthal-Getoor index α requires a faster convergence Δ_n to zero. This is in line with the intuition that when the intensity of small jumps increases (i.e. α increases) more and more frequent observations are needed to have a sufficient performance of the jump filter.

Outline

Jump filtering

Proposition (jump filtering: finite activity)

(i) *without any assumption on the way that* $\Delta_n \to 0$ *as* $n \to \infty$,

$$\frac{1}{n\Delta_n} \sup_{\theta\in\Theta} \left| \int_0^{t_n} f(\theta, X_s)\, dX_s^c - \sum_{i=1}^n f(\theta, X_{t_{i-1}}) \Delta_i^n X \mathbf{1}_{|\Delta_i^n X| \le v_n} \right| \xrightarrow{P} 0$$

(ii) *if* $n\Delta_n^{3-\varepsilon} \to 0$, $\sqrt{n}\Delta_n^{1-\varepsilon/2} \left(\int_{|z|\le 2v_n} \nu(dz) \right)^{1-\varepsilon/2} \to 0$ *and* $\sqrt{n}\Delta_n^{1/2} \int_{|z|\le 2v_n} |z|\nu(dz) \to 0$ *as* $n \to \infty$, *then for any* $\theta \in \Theta$,

$$\frac{1}{\sqrt{n\Delta_n}} \left| \int_0^{t_n} f(\theta, X_s)\, dX_s^c - \sum_{i=1}^n f(\theta, X_{t_{i-1}}) \Delta_i^n X \mathbf{1}_{|\Delta_i^n X| \le v_n} \right| \xrightarrow{P} 0.$$

The case of infinite activity is treated in the following proposition.

Proposition (jump filtering: infinite activity)

Suppose that L is of infinite activity and Assumptions on jumps hold.

(i) *Statement (i) of the previous Proposition holds;*

(ii) *if $n\Delta_n^{3-\varepsilon} \to 0$,*

$$\sqrt{n\Delta_n}\left(\int_{|z|\leq 3v_n/\gamma_{min}} |z|\nu(dz)\right)^{1-\varepsilon/2} \to 0$$

and

$$\sqrt{n}\Delta_n^{3/2-\varepsilon}\left(\int_{|z|\geq 3v_n/\gamma_{min}} \nu(dz)\right)^{1-\varepsilon/2} \to 0$$

as $n \to \infty$, then for any $\theta \in \Theta$, the convergence (ii) of the previous Proposition holds.

Lemme (Euler scheme)

(i) *as* $n \to \infty$,

$$\sup_{\theta \in \Theta} \frac{1}{n\Delta_n} \left| \int_0^{t_n} f(\theta, X_s)\, dX_s^c - \sum_{i=1}^{n} f(\theta, X_{t_{i-1}}) \Delta_i^n X^c \right| \xrightarrow{P} 0;$$

(ii) *if* $n\Delta_n^{3-\varepsilon} \to 0$, *then, as* $n \to \infty$, $\forall \theta \in \Theta$

$$\frac{1}{\sqrt{n\Delta_n}} \left| \int_0^{t_n} f(\theta, X_s)\, dX_s^c - \sum_{i=1}^{n} f(\theta, X_{t_{i-1}}) \Delta_i^n X^c \right| \xrightarrow{P} 0.$$

Outline

Cox-Ingersoll-Ross (CIR) processes with jumps

Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda = 1$ and $(Z_i)_{i\in\mathbb{N}}$ are i.i.d. exponential with rate η, independents of N,

$$L_t = \sum_{i=1}^{N_t} Z_i, \text{ for } t \geq 0.$$

CIR process (X_t) is defined by

$$X_t = (\theta_1 - \theta_2 X_t)\, dt + \sigma\sqrt{X_t}\, dW_t + dL_t,$$

where $\theta_1, \theta_2, \sigma > 0$,

FMLE is given as the solution $\hat{\theta}_n^{\mathsf{CIR}} = (\hat{\theta}_{1,n}^{\mathsf{CIR}}, \hat{\theta}_{2,n}^{\mathsf{CIR}})$ to the following set of linear equations in the parameters θ_1 and θ_2.

$$\theta_1 = \frac{\theta_2 t_n - \sum_{i=1}^n X_{t_i}^{-1} \Delta_i^n X \mathbf{1}_{|\Delta_i^n X| \leq v_n}}{I_n(X, -1)},$$
$$\theta_2 = \frac{\theta_1 t_n - \sum_{i=1}^n \Delta_i^n X \mathbf{1}_{|\Delta_i^n X| \leq v_n}}{I_n(X, 1)},$$

where

$$I_n(X,p) := \sum_{i=1}^n X_{t_i}^p \Delta_i^n Id \quad \text{for } p \in \mathbb{R}.$$

We obtain for $\hat{\theta}_{2,n}^{\mathsf{CIR}}$ the FMLE

$$\hat{\theta}_{2,n}^{\mathsf{CIR}} = \frac{\left(\sum_{i=1}^n X_{t_i}^{-1} \Delta_i^n X \mathbf{1}_{|\Delta_i^n X| \leq v_n} - I_n(X,-1) \sum_{i=1}^n \Delta_i^n X \mathbf{1}_{|\Delta_i^n X| \leq v_n}\right)}{(I_n(X,-1) I_n(X,1) - t_n)}.$$

		$\sigma = 0.25$			$\sigma = 0.5$		
t_n	n	mean	std dev	jumps filt	mean	std dev	jumps filt
5	200	1.7	0.22	6.8	1.7	0.28	8.0
	400	1.9	0.12	5.1	1.8	0.2	6.6
	800	2.0	0.09	4.5	1.9	0.17	5.6
10	500	1.7	0.15	12	1.7	0.21	15
	1000	1.9	0.08	9.7	1.8	0.14	12
	1500	1.9	0.06	9.5	1.9	0.13	11
20	1000	1.8	0.13	25	1.6	0.16	30
	2000	1.9	0.06	19	1.8	0.11	24
	3000	2.0	0.04	19	1.9	0.09	22

Table : Monte Carlo estimates of mean and standard deviation of $\hat{\theta}_{2,n}^{\mathsf{CIR}}$ for a CIR process with Gaussian component and compound Poisson jumps with intensity $\lambda = 1$ and true drift parameter $\theta_2 = 2$.

Hyperbolic diffusions with jumps

$$dX_t = -\frac{\theta X_t}{(1+X_t^2)^{1/2}}\, dt + \sigma\, dW_t + dL_t, \quad X_0 = x.$$

Where $\theta > 0$ and $\sigma > 0$ are unknown and we aim at estimating θ.
$(L_t)_{t\geq 0}$ is an α-stable process with Lévy-Khintchine triplet $(0,0,\nu)$ with $\nu(dx) = dx/|x|^{1+\alpha}$.
Discretization and jump filtering leads to :

$$\hat{\theta}_n^{\mathsf{hyp}} = -\sum_{i=1}^{n} \frac{X_{t_i}}{(1+X_{t_i}^2)^{1/2}} \Delta_i^n X \mathbf{1}_{|\Delta_i^n X| \leq v_n} \left(\sum_{i=1}^{n} \frac{X_{t_i}^2}{(1+X_{t_i}^2)} \right)^{-1}$$

		$\alpha = 0.5$			$\alpha = 1$		
t_n	n	mean	std dev	jumps filt	mean	std dev	jumps filt
5	600	1.7	0.53	26	1.6	0.62	37
	1200	1.9	0.54	27	1.8	0.60	40
	1500	1.9	0.57	26	1.9	0.66	41
10	1000	1.6	0.33	51	1.5	0.40	71
	2000	1.8	0.34	53	1.7	0.38	79
	4000	1.9	0.35	50	1.9	0.43	85
20	2000	1.6	0.23	104	1.6	0.27	142
	4000	1.8	0.24	106	1.7	0.28	158
	8000	1.9	0.23	101	1.9	0.30	170

Table : Monte Carlo estimates of mean and standard deviation from 500 samples of $\hat{\theta}_n^{\mathsf{hyp}}$ for a hyperbolic diffusion process with Gaussian component and α-stable jumps and true drift parameter $\theta = 2$

Outline

Conclusion

our work shows that by focusing on the drift estimation the condition $n\Delta_n^2 \to 0$ can be relaxed.

is in accordance with the condition $n\Delta_n^3 \to 0$ of [Florens-Zimrou(89)] and [Yoshida(92)] in the case of drift estimation for continuous diffusions.

Outline

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Thank you for your attention!