Parameter estimation of Ornstein-Uhlenbeck process generating a stochastic graph

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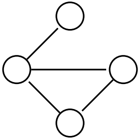
Overview

1. Stochastic graphs
2. Mixing properties for multidimensional Ornstein-Uhlenbeck processes
3. Convergence of statistics of binary observations

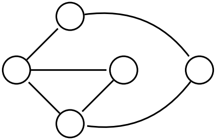
Stochastic graphs

Typical graph evolution (preferential attachment)

Step n $\longrightarrow$ Step $n+1$



Node/Edge
addition/deletion



Evolution in **continuous time**?

Ornstein-Uhlenbeck processes

Definition: Ornstein-Uhlenbeck process X

Take $A \in \mathcal{M}_{d,d}(\mathbb{R})$ and $\Sigma \in \mathcal{M}_{d,q}(\mathbb{R})$ where $d, q \in \mathbb{N}^+$, $(W_t)_{t \in \mathbb{R}^+}$ a q-dimensional Brownian motion with respect to $\mathcal{F}$. $X = (X_t : t \geq 0)$ is an Ornstein-Uhlenbeck process when it solves

$$\mathrm{d}X_t = -AX_t \mathrm{d}t + \Sigma \mathrm{d}W_t, \quad X_0 \text{ given.} \tag{1}$$

Standing assumptions

- $\Sigma\Sigma^*$ is invertible
- $a_0 := \min_{\lambda \in \mathrm{Sp}(A)} \mathcal{R}e(\lambda) > 0$
- $X_0 \overset{\mathrm{d}}{=} \mathcal{N}(0, V_\infty)$, $V_t = \int_0^t e^{-Au} \Sigma\Sigma^* e^{-A^* u} \mathrm{d}u$

X is **ergodic** and **stationary**.

Graph generation

Definition: Graph observation Y

Take S a measurable subset of $\mathbb{R}^d$. Define:

$$Y_t^S = \mathbb{1}_{X_t \in S} \tag{2}$$

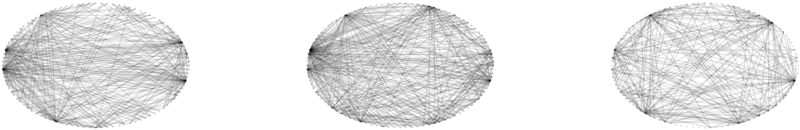
Take for example $S^{ij} := \{x : x^i \geq 1, x^j \geq 1\}$ and $Y_t^{ij} := Y_t^{S^{ij}}$. Then Y_t is a graph.

A, Σ	Y_t
Represents underlying relations	Represents observed relations
Stable	Evolves in time

Interbank lending model

The preceding is inspired by a model of interbank lending [CFS15, FI13]:

$$\mathrm{d}X_t^i = -\frac{a}{D}\sum_{j=0}^{D}\left(X_t^i - X_t^j\right)\mathrm{d}t + \sigma^i \mathrm{d}W_t^i$$



Problematic

How to estimate $(A, \Sigma\Sigma^*)$ from the observation of Y?

- Use long time limit ($n\Delta_n \to +\infty$) to apply ergodic properties (estimate V_∞)
- Use high frequency ($\Delta_n \to 0$) to estimate parameters related to local fluctuations (estimate $\Sigma\Sigma^*$)

Related: estimation from sign changes [Flo87], estimation from thresholded process [IUY09]

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Gebelein inequality

Theorem ([Jan97, Theorem 10.11])

Take H, K two closed subspaces of some Gaussian Hilbert space. Define P_{HK} the restriction to H of the orthogonal projection onto K. Define the maximal correlation coefficient between variables A, B respectively measurable w.r.t. the sigma field generated by H and K:

$$\rho(H,K) = \sup_{A \in L^2(H), B \in L^2(K)} \left| \mathbb{C}\mathrm{or}\left(A,B\right) \right|.$$

Then we have:

$$\rho(H,K) = \|P_{HK}\|$$

where $\|\cdot\|$ is the operator norm.

In practice, it means that

$$\mathbb{C}\mathrm{ov}\left(f(X), g(Y)\right) \leq \rho(X,Y)\sqrt{\mathbb{V}\mathrm{ar}\left(f(X)\right)\mathbb{V}\mathrm{ar}\left(g(Y)\right)}$$

Gebelein inequality for Ornstein-Uhlenbeck processes

Proposition

Take (X, Y) a Gaussian vector. Assume that $\mathbb{C}\mathrm{ov}\,(X)$, $\mathbb{C}\mathrm{ov}\,(Y)$ are non-degenerate. Then we have

$$\rho(X, Y) = \|\mathbb{C}\mathrm{ov}\,(X)^{-1/2}\,\mathbb{C}\mathrm{ov}\,(X, Y)\,\mathbb{C}\mathrm{ov}\,(Y)^{-1/2}\,\|.$$

For stationary OU processes, we have:

$$\mathbb{C}\mathrm{ov}\,(X_s) = \mathbb{C}\mathrm{ov}\,(X_t) = V_\infty, \qquad \mathbb{C}\mathrm{ov}\,(X_t, X_s) = e^{-A(t-s)}V_\infty.$$

Therefore, with $v_M = \max_{\lambda \in \mathrm{Sp}(V_\infty)} \lambda$, $v_m = \min_{\lambda \in \mathrm{Sp}(V_\infty)} \lambda$:

$$\rho(X_s, X_t) = \|V_\infty^{-1/2} e^{-A(t-s)} V_\infty^{1/2}\| \leq \sqrt{\frac{v_M}{v_m}} e^{-a_0|t-s|}.$$

Correlation inequality for Ornstein-Uhlenbeck processes

Theorem (Mixing properties)

There exists a finite constant $C_{(3)}$, depending only on V_∞, such that for any $t \geq s \geq 0$ and functions φ, ϕ, square-inegrable w.r.t. the law of X:

$$\left|\mathbb{C}\mathrm{ov}\left(\varphi\left((X_u)_{u\leq s}\right), \phi\left((X_v)_{v\geq t}\right)\right)\right| \leq C_{(3)} e^{-a_0|t-s|} \sqrt{\mathbb{V}\mathrm{ar}\left(\varphi\left((X_u)_{u\leq s}\right)\right) \mathbb{V}\mathrm{ar}\left(\phi\left((X_v)_{v\geq t}\right)\right)}. \quad (3)$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\varphi_s \phi_t\right] &= \mathbb{E}\left[\varphi_s \mathbb{E}\left[\phi_t \mid X_s\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\varphi_s \mid X_s\right] \mathbb{E}\left[\phi_t \mid X_s\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\varphi_s \mid X_s\right] \phi_t\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\varphi_s \mid X_s\right] \mathbb{E}\left[\phi_t \mid X_t\right]\right] \\
&\leq \rho(X_s, X_t) \sqrt{\mathbb{V}\mathrm{ar}\left(\mathbb{E}\left[\varphi_s \mid X_s\right]\right) \mathbb{V}\mathrm{ar}\left(\mathbb{E}\left[\phi_t \mid X_t\right]\right)} \\
&\leq \rho(X_s, X_t) \sqrt{\mathbb{V}\mathrm{ar}\left(\varphi_s\right) \mathbb{V}\mathrm{ar}\left(\phi_t\right)}.
\end{aligned}$$

Bound on variance of sums of functionals of X

Corollary

Consider a measurable function $g : \mathbb{N} \times \mathbb{N} \times \mathcal{C}^0([0,1], \mathbb{R}^d) \to \mathbb{R}$ *such that* $\mathbb{E}\left[g(k, n, (X_s)_{k\Delta_n \le s \le (k+1)\Delta_n})^2\right] < +\infty$ *for any* $k, n \in \mathbb{N}$*. For* $n \in \mathbb{N}$ *define*

$$v_n^2 = \sup_{k<n} \mathbb{V}\mathrm{ar}\left(g(k, n, (X_s)_{k\Delta_n \le s \le (k+1)\Delta_n})\right),$$

$$\xi_k^{(n)} = \sqrt{\frac{\Delta_n}{n}} g(k, n, (X_s)_{k\Delta_n \le s \le (k+1)\Delta_n}).$$

Then, there is a finite constant $C_{(4)}$*, dependent only on the parameters* A, Σ *of the model, such that:*

$$\mathbb{V}\mathrm{ar}\left(\sum_{k=0}^{n-1} \xi_k^{(n)}\right) \le C_{(4)} v_n^2. \tag{4}$$

Bound on variance of sums of functionals of X

Proof.

Denote $g_k = g(k, n, (X_s)_{s\in I(k)})$ where $I_n(k) = [k\Delta_n, (k+1)\Delta_n]$.

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left(\sum_{k=0}^{n-1}\xi_k^{(n)}\right) &= \frac{\Delta_n}{n}\sum_{k=0}^{n-1}\mathbb{V}\mathrm{ar}\left(g_k\right) + \frac{2\Delta_n}{n}\sum_{k=0}^{n-1}\sum_{l=k+1}^{n-1}\mathbb{C}\mathrm{ov}\left(g_k, g_l\right) \\
&\leq \frac{\Delta_n}{n} n v_n^2 + \frac{2\Delta_n}{n} n \sum_{m\geq 0} C_{(3)} v_n^2 e^{-a_0 m\Delta_n} \\
&\leq v_n^2\left(\Delta_n + 2C_{(3)}\frac{\Delta_n}{1-e^{-a_0\Delta_n}}\right) \\
&\leq C_{(4)} v_n^2
\end{aligned}$$

Where we use that for $l > k$, $u \in I_n(k), v \in I_n(l)$, we have $u \leq (k+1)\Delta_n \leq l\Delta_n \leq v$, and apply Theorem 2:

$$\mathbb{C}\mathrm{ov}\left(g_k, g_l\right) \leq C_{(3)} e^{-a_0|k+1-l|\Delta_n}\sqrt{\mathbb{V}\mathrm{ar}\left(g_k\right)\mathbb{V}\mathrm{ar}\left(g_l\right)}. \qquad (5)$$

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Occupation time convergence

Definition

The occupation time statistic is defined as:

$$\mathrm{OT}_n^S = \frac{1}{n}\sum_{k=0}^{n-1} Y_{k\Delta_n}^S = \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{X_{k\Delta_n}\in S}. \tag{6}$$

$$\mathrm{OT}_n^S = \sum_{k=0}^{n-1} \sqrt{\frac{\Delta_n}{n}} \frac{\mathbb{1}_{X_{k\Delta_n}\in S}}{\sqrt{n\Delta_n}}$$

$$\mathbb{V}\mathrm{ar}\left(\frac{\mathbb{1}_{X_{k\Delta_n}\in S}}{\sqrt{n\Delta_n}}\right) \propto \frac{1}{n\Delta_n}$$

Using Corollary 1, we have $\mathbb{V}\mathrm{ar}\left(\mathrm{OT}_n^S\right) = O\left(n^{-1}\Delta_n^{-1}\right)$ and convergence in L^2 under the assumption $n\Delta_n \to +\infty$.

Crossing number convergence

Definition

We define the crossings statistic by:

$$\mathcal{C}_n^S = \frac{1}{n\sqrt{\Delta_n}} \sum_{k=0}^{n-1} \mathbb{1}_{Y^S_{k\Delta_n} \neq Y^S_{(k+1)\Delta_n}}. \tag{7}$$

We choose here $S = \{x^1 \geq 1\}$ and consider only crossings from 0 to 1. Write $Z_k^{(n)} = \mathbb{1}_{X^1_{k\Delta_n} < 1} \mathbb{1}_{X^1_{(k+1)\Delta_n} \geq 1}$.

$$\frac{1}{n\sqrt{\Delta_n}} \sum_{k=0}^{n-1} Z_k^{(n)} = \sum_{k=0}^{n-1} \sqrt{\frac{\Delta_n}{n}} \frac{Z_k^{(n)}}{\sqrt{n}\Delta_n}, \quad \text{and } \mathbb{E}\left[Z_k^{(n)}\right] \sim \mathbb{V}\mathrm{ar}\left(Z_k^{(n)}\right) \sim \sqrt{\Delta_n}$$

$$\mathbb{V}\mathrm{ar}\left(\frac{Z_k^{(n)}}{\sqrt{n}\Delta_n}\right) = O\left(\frac{1}{n\Delta_n^{3/2}}\right)$$

Expectations and CLT

We have $\mathbb{E}\left[\mathrm{OT}_n^S\right] = \nu_\infty(S)$. Assuming $n\Delta_n \to +\infty$,

$$\mathrm{OT}_n^S \xrightarrow{L^2} \nu_\infty(S)$$

We also have:

Theorem

$$\sqrt{n\Delta_n}\left(\mathrm{OT}_n^{[1,+\infty[} - \nu_\infty\left([1,+\infty[\right)\right) \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \nu_\infty\left(\sigma^2 F'^2\right)\right) \qquad (8)$$

as $n \to +\infty$, where F solves $LF + (\mathbb{1}_{x\geq 1} - \nu_\infty([1,+\infty[)) = 0$ with L the infintesimal generator of the OU.

We have $\mathbb{E}\left[Z_k^{(n)}\right] \sim \sqrt{\Delta_n}\sqrt{\frac{(\Sigma\Sigma^*)^{11}}{2\pi}}\mu_{V_\infty^{11}}(1)$. Assuming $n\Delta_n^{3/2} \to +\infty$,

$$\mathcal{C}_n \xrightarrow{L^2} 2\sqrt{\frac{(\Sigma\Sigma^*)^{11}}{2\pi}}\mu_{V_\infty^{11}}(1)$$

Application

How to estimate $(A, \Sigma\Sigma^*)$ from the observation of Y?

Assume we observe Y^{ij} for $S^{ij} = \{x^i \geq 1, x^j \geq 1\}$.
Assume $A = diag(a_1, \ldots a_d)$. Then

$$V_\infty^{ij} = \frac{(\Sigma\Sigma^*)^{ij}}{a_i + a_j}.$$

- Using $\mathcal{C}_n^{S^{ii}}$, we can estimate $(\Sigma\Sigma^*)^{ii}$
- Using $\mathrm{OT}_n^{S^{ii}}$, we can estimate V_∞^{ii}, and we get a_i
- Using $\mathrm{OT}_n^{S^{ij}}$, we can estimate V_∞^{ij}, and we get $(\Sigma\Sigma^*)^{ij}$

Short references

D. Florens-Zmirou.
Estimation du paramètre d'une diffusion par les changements de signe de sa discrétisée.
Comptes Rendus de l'Académie des Sciences Paris, 305:661 – 664, 1987.

S. Janson.
Gaussian Hilbert Spaces.
Cambridge Tracts in Mathematics. Cambridge University Press, 1997.

S.M. Iacus, M. Uchida, and N. Yoshida.
Parametric estimation for partially hidden diffusion processes sampled at discrete times.
Stochastic Processes and their Applications, 119(5):1580 – 1600, 2009.

R. Carmona, J.-P. Fouque, and L.-H. Sun.
Mean field games and systemic risk.
Commun. Math. Sci., 13(4):911–933, 2015.

J.-P. Fouque and T. Ichiba.
Stability in a model of interbank lending.
SIAM J. Financial Math., 4(1):784–803, 2013.

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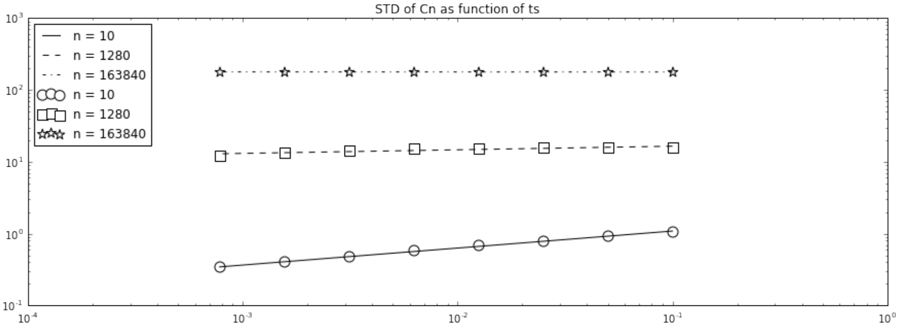
Optimality of $\mathcal{C}_n$ convergence speed?



$$\sqrt{\mathbb{V}\mathrm{ar}\left(\sum_k Z_k^{(n)}\right)} \propto \Delta_n^0$$

Proof of CLT for OT_n

$$\mathrm{OT}_t^c = \frac{1}{t}\int_0^t \mathbb{1}_{X_s \geq 1} \mathrm{d}s$$

$$\int_0^t \hat{f}(X_s)\mathrm{d}s = t\left(\mathrm{OT}_t^c - \nu_\infty\left([1,+\infty[\right)\right) \quad \hat{f}(x) = f(x) - \nu_\infty\left([1,+\infty[\right)$$

$$L = -ax\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} \qquad LF = -\hat{f}$$

$$M_t = F(X_t) - F(X_0) + \int_0^t \hat{f}(X_s)\mathrm{d}s = \int_0^t \sigma F'(X_s)\mathrm{d}W_s$$

$$\frac{M_t}{\sqrt{t}} = \frac{F(X_t) - F(X_0) + \left(\mathrm{OT}_t^c - \nu_\infty\left([1,+\infty[\right)\right)t}{\sqrt{t}} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \nu_\infty\left(\sigma^2 F'^2\right)\right)$$

Proof of CLT for OT_n

$$\frac{\mathrm{OT}_t^c}{\sqrt{t}} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \nu_\infty\left(\sigma^2 F'^2\right)\right)$$

$$\begin{aligned} D_n &:= \sqrt{n\Delta_n}\left(\mathrm{OT}_n^{[1,+\infty[} - \mathrm{OT}_{n\Delta_n}^c\right) \\ &= \sqrt{\frac{\Delta_n}{n}} \sum_{k=0}^{n-1} \int_0^{\Delta_n} \frac{f\left(X_{k\Delta_n}\right) - f\left(X_{k\Delta_n+u}\right)}{\Delta_n} \mathrm{d}u \end{aligned}$$

$$\sqrt{n\Delta_n}\left(\mathrm{OT}_n^{[1,+\infty[} - \nu_\infty\left([1,+\infty[\right)\right) \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \nu_\infty\left(\sigma^2 F'^2\right)\right)$$