Spatial Mixing Properties of Random Tessellations

Werner Nagel, Jena

joint work with Servet Martínez, Santiago de Chile

Outline of this lecture

- Ergodicity and mixing properties for random sets in euclidean space
- Poisson hyperplane tessellations
- Poisson Voronoi tessellations
- STIT tessellations

Motivation

Ergodic and mixing properties

- express different levels of weak stochastic dependencies,
- express 'long or short distance' dependencies (in space)
- provide sufficient conditions for limit theorems

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Intuitive interpretation:

Ergodicity:

'All essential features of a random structure can be observed in a single realization.' (if the observation window is large enough)

Tail triviality:

'The behavior of a random structure in its tail does not depend on its behavior in any bounded part.'

E.g. The event 'There are infinitely many triangles in a tessellation' has probability either 1 or 0.

Mixing:

'The dependencies between parts of a random structure in two distant regions of space vanish with growing distance.'

Very rough survey

independence $\Rightarrow$ ... $\Rightarrow$ β-mixing $\Rightarrow$ α-mixing $\Rightarrow$ tail triviality $\Rightarrow$ ergodic-mixing $\Rightarrow$ ergodic $\Rightarrow$ ...

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For sequences of random variables:

Bradley, R.C. (2005) Basic Properties of Strong Mixing Conditions. A Survey and Some Open Questions. *Probability Surveys* **2**, 107–144.

Bradley, R.C. (2007) *Introduction to Strong Mixing Conditions.* Vol I–III.

Ergodicity and mixing in Euclidean Space

Stationary random measure or point process
or stationary random closed set

- ergodicity (Nguyen/Zessin for marked PP and Boolean models, 1979)
- ergodic-mixing (Daley/Vere-Jones 1988 for random measures)
- tail triviality (Daley/Vere-Jones 1988 for random measures)
- β-mixing (Heinrich 1994)

(For more biographical remarks and references see Heinrich et al. and Schneider/Weil, chapter 9)

Ergodicity and mixing in euclidean space

Consider random closed sets (RACS)

$\mathcal{F}$ . . . set of all closed subsets of $\mathbb{R}^d$,

$\mathcal{B}(\mathcal{F})$ . . . (Borel) σ-algebra on $\mathcal{F}$,

$\mathcal{C}$ . . . set of all compact subsets of $\mathbb{R}^d$

Events to be considered:

$\{T \in \mathcal{F} : \; T \cap C = \emptyset\}, \quad C \in \mathcal{C}$

Ergodicity and mixing in euclidean space

σ-algebras to be considered:

Ergodicity and mixing in euclidean space

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- σ-algebra of translation-invariant events $\mathcal{T}_{inv}$,

$$A \in \mathcal{B}(\mathcal{F}) : \quad A \quad (A+h) \quad , \forall h \in \mathbb{R}^d$$

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- $\mathcal{T}_{tail}$ ... σ-algebra of terminal events, the tail σ-algebra

- pairs $\mathcal{T}(W'), \mathcal{T}(W^c)$ for windows $W' \subset W$ with

  $\mathcal{T}(W') = \sigma\left(\{\{T \in \mathcal{F} : T \cap C = \emptyset\} : C \subset W', C \in \mathcal{C}\}\right),$

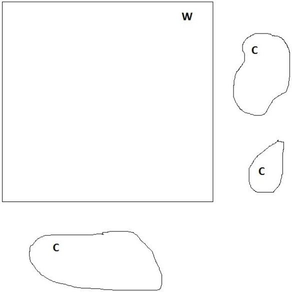
  $\mathcal{T}(W^c) = \sigma\left(\{\{T \in \mathcal{F} : T \cap C = \emptyset\} : C \subset W^c, C \in \mathcal{C}\}\right)$

Ergodicity and mixing in euclidean space

Events which are determined by the behavior outside a window W.

σ-algebra:

$$\mathcal{T}(W^c) = \sigma\left(\{\{T \in \mathcal{F} : T \cap C = \emptyset\} : C \subset W^c, C \in \mathcal{C}\}\right).$$



Ergodicity and mixing in euclidean space

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Ergodicity and mixing in euclidean space

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Definition:

The tail-σ-algebra (of terminal events) on $\mathbb{T}$ is defined as

$$\mathcal{T}_{tail} = \bigcap_{n=1}^{\infty} \mathcal{T}(W_n^c)$$

with $W_n = [-n, n]^d, \; n \in \mathbb{N}$. □

Ergodicity and mixing in euclidean space

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Examples of terminal events:

"there are infinitely many triangles in the tessellation", or

"there are infinitely many cells in the tessellation with inradius > 1"

Ergodicity and mixing in euclidean space

Let Y be a stationary (homogeneous) random closed set in $\mathbb{R}^d$.

Definition: Y is

- ergodic, if $P(Y \in A) \in \{0, 1\}, \qquad \forall A \in \mathcal{T}_{inv},$

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$$\lim_{||h|| \to \infty} P(Y \cap C_1 = \emptyset, Y \cap (C_2 + h) = \emptyset)$$

$$= P(Y_t \cap C_1 = \emptyset) \cdot P(Y_t \cap C_2 = \emptyset),$$

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- tail-trivial, if $P(Y \in A) \in \{0,1\}, \qquad \forall A \in \mathcal{T}_{tail}$. $\square$

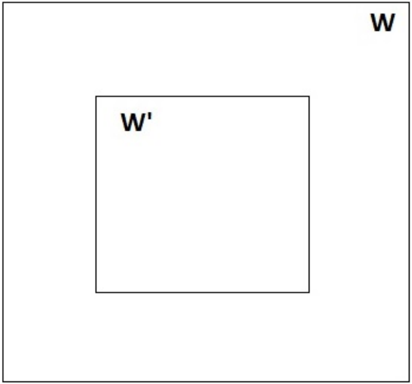
Strong mixing in euclidean space

Behavior inside a window $W' \subset W$

$$\mathcal{T}(W') = \sigma\left(\{\{T \in \mathcal{F} : T \cap C = \emptyset\} : C \subset W', \ C \in \mathcal{C}\}\right).$$

Behavior outside a window W

$$\mathcal{T}(W^c) = \sigma\left(\{\{T \in \mathcal{F} : T \cap C = \emptyset\} : C \subset W^c, \ C \in \mathcal{C}\}\right).$$



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$$\alpha(a, b) := \sup |P(A \cap B) - P(A)P(B)|,$$
$$A \in \mathcal{T}(W'),\ B \in \mathcal{T}(W^c),$$

$$\beta(a, b) := \frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)|,$$

where the supremum is taken over all pairs of finite partitions of $\mathcal{F}$:
$\{A_i, i = 1, \ldots I\}$ for events $A_i \in \mathcal{T}(W')$
and
$\{B_j, j = 1, \ldots J\}$ with $I, J \in \mathbb{N}$ for events $B_j \in \mathcal{T}(W^c)$.

Definition: A stationary (homogeneous) random closed set Y in $\mathbb{R}^d$ is

- α-mixing, if $\forall a > 0 : \lim_{b \to \infty} \alpha(a, b) = 0,$
- β-mixing/absolutely regular, if $\forall a > 0 : \lim_{b \to \infty} \beta(a, b) = 0.$

Strong mixing in euclidean space

Equivalent to β-mixing:

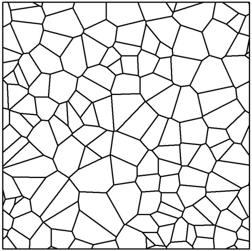
$\forall \varepsilon > 0, \forall a > 0$ $\exists D \in \mathcal{T}([-a,a]^d)$ with $P(D) \geq 1-\varepsilon$, $\exists b > a$, such that

$$\forall A \in \mathcal{T}([-a,a]^d), \forall B \in \mathcal{T}(([-b,b]^d)^c) : \; A \subseteq D, \; P(A) > 0$$

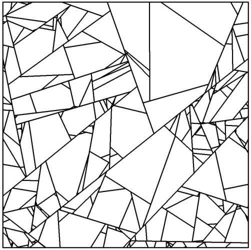
$$\Rightarrow |P(B|A) - P(A)| = \frac{|P(A \cap B) - P(A)P(B)|}{P(A)} \leq \varepsilon$$

Random tessellations

Three reference models



Poisson-Voronoi



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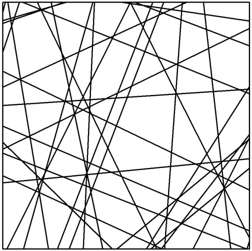

Poisson line

Ergodicity and mixing in euclidean space

$\mathbb{T}$ ... the set of all tessellations of $\mathbb{R}^d$

In this lecture:

A tessellation is considered as the closed set of its cell boundaries.

$$\mathbb{T} \subset \mathcal{F}$$

Poisson hyperplane tessellations

Theorem: (Schneider and Weil, Theorem 10.5.3)

A Poisson hyperplane tessellations is ergodic-mixing if the directional distribution has zero mass on all great subspheres of $\mathcal{S}^{d-1}$.

In $\mathbb{R}^2$: A Poisson line tessellations is ergodic-mixing if there are a.s. no pairs of parallel lines.

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Martínez/N. (2012): For Poisson hyperplane tessellations:
The tail-σ-algebra is not trivial.

Example: For the stationary and isotropic case the event 'There is a hyperplane that intersects the unit ball B_1 centered at 0' belongs to the tail-σ-algebra and its probability is neither 0 nor 1.

Poisson-Voronoi tessellations are strongly mixing

Theorem (Heinrich 1994)

For a stationary point process (PP) in $\mathbb{R}^d$ and the generated Voronoi tessellation (VT)

$$\beta_{VT}(a, b) \leq \beta_{PP}(a, b) + R(a, b)$$

where $R(a, b)$ is the probability for a set of certain point configurations.

Poisson-Voronoi tessellations are strongly mixing

Windows $W' = [-a, a]^d$, $W = [-b, b]^d$, $0 < a < b$.

Theorem (Heinrich 1994)

For a stationary Poisson point process with intensity λ in $\mathbb{R}^d$ and the generated Poisson-Voronoi tessellation (PVT)

$$\beta_{PVT}(a, b) \leq$$

$$\begin{cases} c_1 d \left(\frac{b-a}{a}\right)^{d-1} \exp\left[-\lambda(2a)^{d-1}(b-a)/24\right] & \textit{if} \quad b - a > c_0 a, \\ c_2 d \left(\frac{a}{b-a}\right)^{d-1} \exp\left[-\lambda(2/c_0)^{d-1}(b-a)^d/24\right] & \textit{if} \quad b - a \leq c_0 a, \end{cases}$$

with explicitly given $c_i(d)$, $i = 0, 1, 2$.

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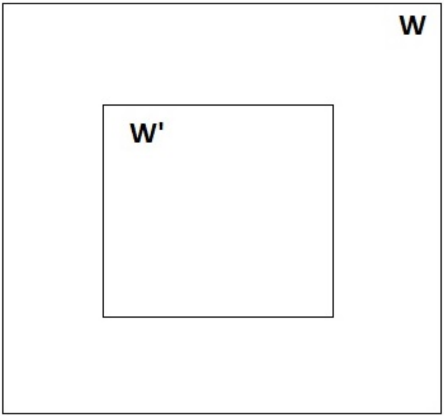

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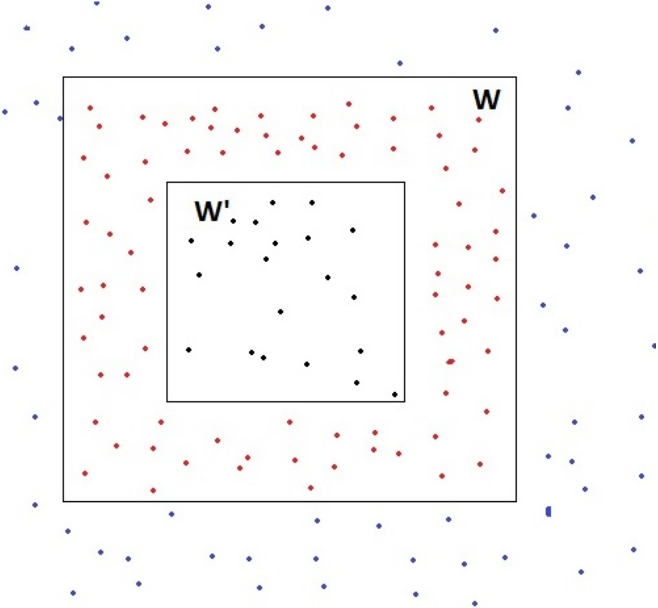

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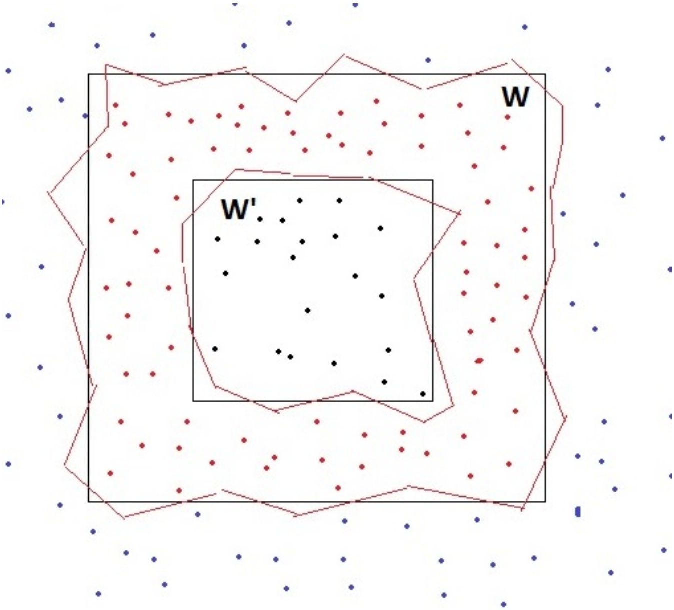

Poisson-Voronoi tessellations

Remarks:

- (Heinrich 1994) The upper bound given in the last Theorem implies $\lim_{b\to\infty} \beta_{PVT}(a,b) = 0$ and, moreover, that the decay of $\beta_{PVT}(a,b)$ is sufficiently strong to derive a CLT for the number of nodes and for the total edge length.

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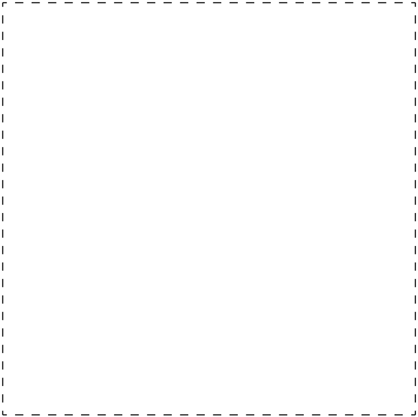
- Calka and Chenavier (*Extremes* 2014) consider order statistics of functionals $f(C)$, such as inradius, circumradius, area, volume of the Voronoi flower, for the cells C of Posson-Voronoi and Poisson-Delaunay tessellations resp. in a bounded Window W. They formulate (still rather involved technical and not yet 'standard' mixing) sufficient conditions for the convergence (for large W) of these order statistics.

STIT tessellations – Construction

Random tessellations generated by sequential cell division



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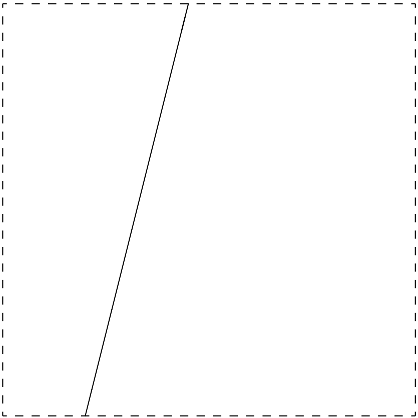

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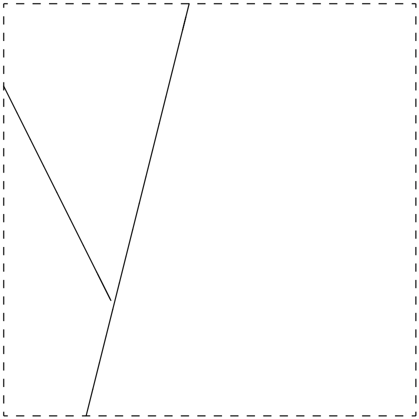

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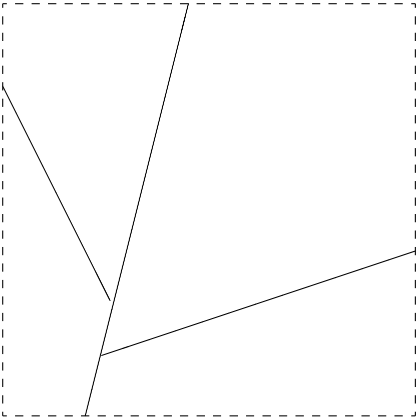

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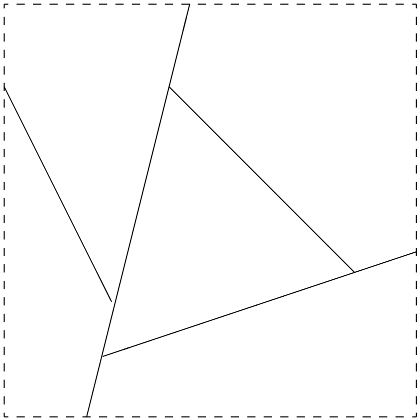

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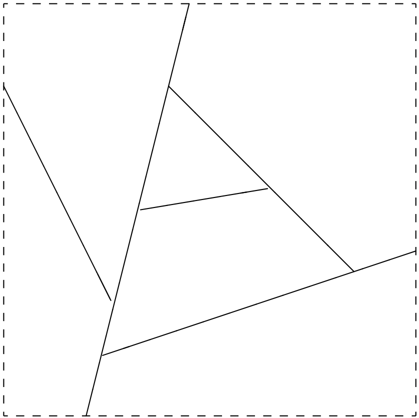

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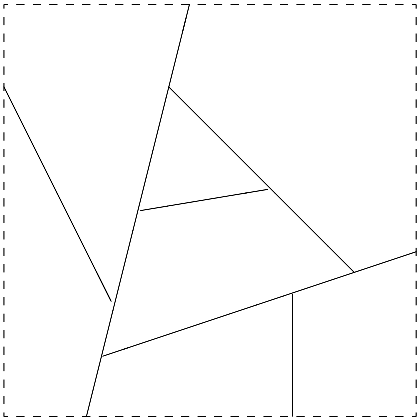

Construction of STIT tessellations

$(Y_t, t > 0)$ ... STIT process in $\mathbb{R}^d$

determined by

- Λ ... translation invariant measure on $(\mathcal{H}, \mathfrak{H})$ on the space of hyperplanes in $\mathbb{R}^d$

  $\Lambda = \text{image}\,[\gamma \cdot \ell \otimes \theta]$;

  $\gamma > 0$ ... intensity,

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- $\Lambda_{[C]} = \frac{1}{\Lambda([C])} \Lambda(\cdot \cap [C])$ ... division rule

Ergodic properties of STIT

Theorem (Lachièze-Rey 2010)

For all $t > 0$, the STIT tessellation Y_t is ergodic-mixing in space, i.e.

for all Borel sets $A, B \subset \mathbb{R}^d$, $h \in \mathbb{R}^d$

$$\lim_{||h|| \to \infty} P(Y_t \cap A = \emptyset, Y_t \cap (B+h) = \emptyset)$$

$$= P(Y_t \cap A = \emptyset) \cdot P(Y_t \cap B = \emptyset)$$

(Y_t as the random closed set of cell boundaries.)

Ergodic-mixing of STIT

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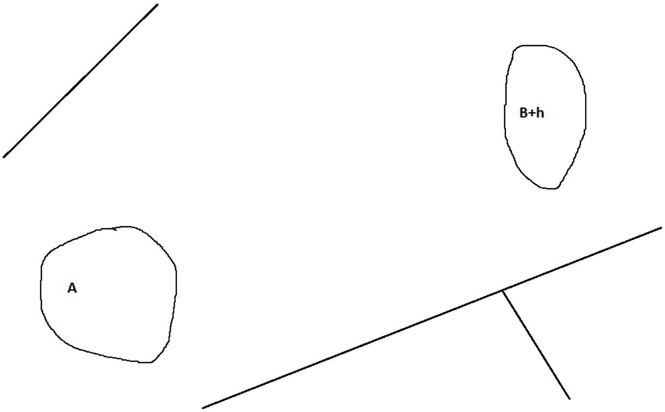

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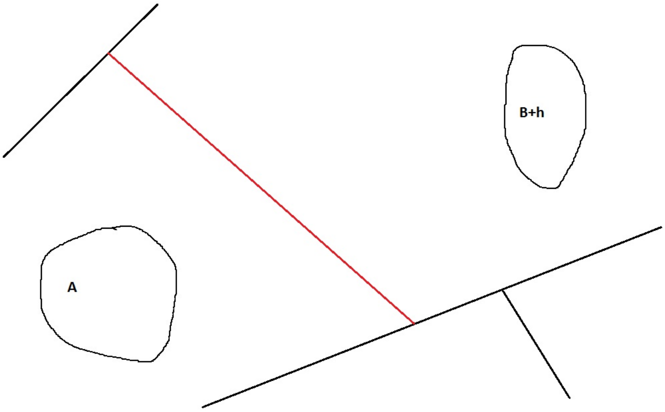
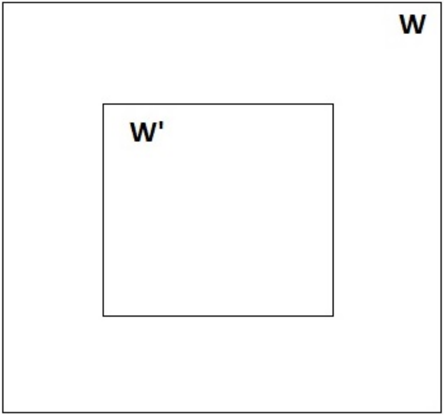
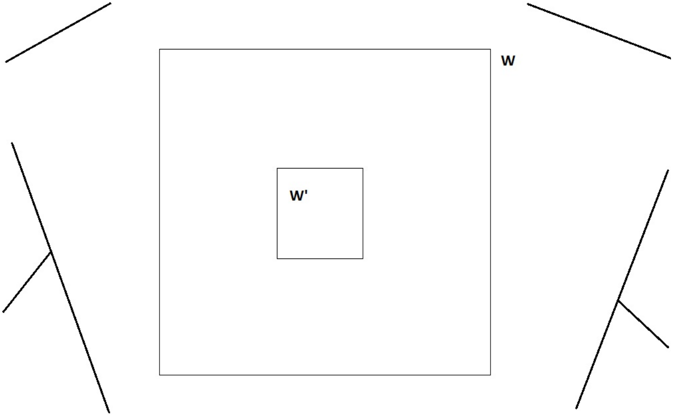


Strong mixing of STIT

Idea: With a probability $> 1 - \epsilon$ appears an encapsulation of W' inside W (before W' is intersected).



Encapsulation for STIT



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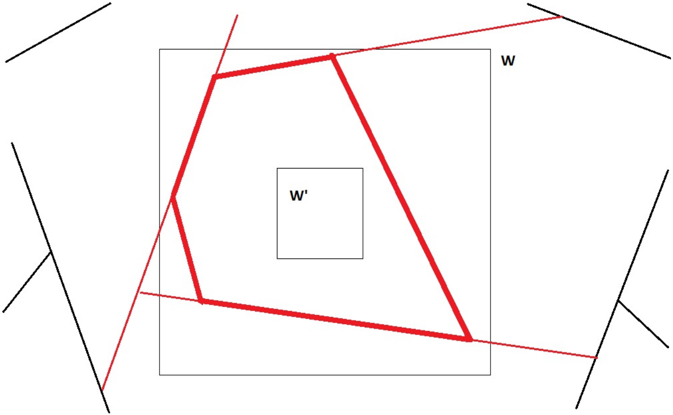


STIT is β-mixing

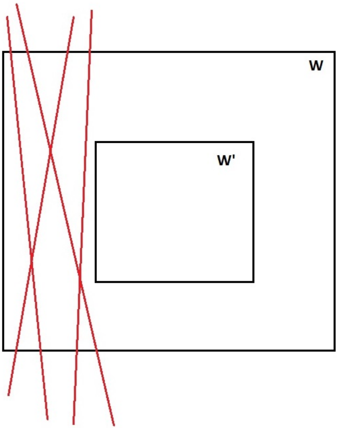
Denote

$[C]$ ... set of hyperplanes that intersect C,

$$\zeta(T \wedge W') = \sum_{cells\ C^i\ in\ T \wedge W'} \Lambda([C^i]),$$

If Λ is rotation invariant and $d = 2$, then
$\zeta(T \wedge W')$ is (up to a constant)
the boundary length of W + $2\times$ the total length of edges of $T \cap W$.

STIT is β-mixing



$[f_j'|f_j]$ ... set of hyperplanes that separate facet f_j' of W' from parallel facet f_j of W,

$$L = \min\{\Lambda([f_j'|f_j]),\ j = 1, ..., d\}.$$

STIT is β-mixing

Theorem (Martínez/N., 2014)

Let $(Y_t,\ t > 0)$ be the STIT tessellation process determined by Λ.
For $0 < a < b$ let $W' = [-a, a]^d \subset W = [-b, b]^d$.
Then for a fixed $t > 0$ and all $0 < s < t,\ M > 0$ we have

$$\beta(a, b) <$$

$$P(\zeta(Y_t \wedge W') \geq M) + P(\zeta(Y_t \wedge W') < M) \times$$

$$\times \left[2 + \mathrm{e}^{sM} - \mathrm{e}^{-sM} - (1 + \mathrm{e}^{-sM})\mathrm{e}^{-s\Lambda([W'])} \left(1 - \mathrm{e}^{-sL(a,b)}\right)^{2d}\right].$$

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$$\beta(a, b) <$$

$$P(\zeta(Y_t \wedge W') \geq M) + P(\zeta(Y_t \wedge W') < M) \times$$

$$\times \left[2 + \mathrm{e}^{sM} - \mathrm{e}^{-sM} - (1 + \mathrm{e}^{-sM}) \mathrm{e}^{-s\Lambda([W'])} \left(1 - \mathrm{e}^{-sL(a,b)}\right)^{2d} \right].$$

This upper bound can now be minimized by choosing appropriate $M > 0$ and $0 < s < t$.

STIT is β-mixing

Theorem (Martínez/N., 2014)

For $t > 0$ let be Y_t the state at time t of a STIT tessellation process determined by the hyperplane measure Λ. Then for $0 < a < b$, $W' = [-a, a]^d \subset W = [-b, b]^d$ and all $\eta \in (0, 1)$ there exists a constant $\kappa = \kappa(t, a, \eta) < \infty$ such that

$$\beta(a, b) \leq \kappa\, b^{-\eta},$$

i.e. STIT is β-mixing.

An Application: Variances for Functionals of STIT

Lemma

(Yoshihara-Heinrich)
For all real valued random variables $X, Y \in L^2(P)$ and all $\delta > 0$

$$|\mathrm{Cov}(X, Y)| \leq 2 \left(\mathbb{E}\left(|X|^{2+\delta} \right) \right)^{\frac{1}{2+\delta}} \left(\mathbb{E}\left(|Y|^{2+\delta} \right) \right)^{\frac{1}{2+\delta}} \left(\beta(\sigma(X), \sigma(Y)) \right)^{\frac{\delta}{2+\delta}}$$

where $\sigma(X)$, $\sigma(Y)$ denote the σ-algebras generated by X and Y respectively.

An Application: Variances for Functionals of STIT

Theorem (Martínez/N., 2015)

Let X be an additive functional such that for some $\delta > 0$ holds $\mathbb{E}\left(X([-1,1]^d \cap Y_t)^{2+\delta}\right) < \infty$.
Then for all $0 < \varepsilon < 1$

$$Var\left(\frac{1}{(2n)^d} X([-n,n]^d \cap Y_t)\right) \leq O\left(n^{-(1-\varepsilon)\frac{\delta}{2+\delta}}\right)$$

as $n \to \infty$.

Examples:

X ... number of vertices in a window, or
X ... total k-volume of k-dimensional faces of cells inside a window

Earlier Results by Schreiber and Thäle (2010/2012)

Y_t stationary and isotropic STIT at time $t > 0$. $W_n = [-n, n]^d$.

- For $d = 2$: $X(W_n \cap Y_t)$ ...
  - number of vertices, or
  - number of center points of maximal ($I-$)segments, or
  - the total length of edges,

  then

$$Var\left(\frac{1}{(2n)^2} X(W_n \cap Y_t)\right) = O\left(n^{-2} \ln n\right) \quad \text{for} \quad n \to \infty.$$

- For $d \geq 3$:
  $X(W_n \cap Y_t)$ ... the total surface area of cell boundaries, then

$$Var\left(\frac{1}{(2n)^d} X(W_n \cap Y_t)\right) = O\left(n^{-2}\right) \quad \text{for} \quad n \to \infty.$$

Hence, in these cases the asymptotic boundaries for the variance are considerably smaller than our upper bound $O\left(n^{-(1-\varepsilon)\frac{\delta}{2+\delta}}\right)$.

Concluding remarks

- Open problem: More precise bounds for the decay of $\beta(a, b)$ for STIT (cf. Heinrich's papers).
- Open problem: Transfer of results by Calka/Chenavier (order statistics/extreme values) to STIT?

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Conjectures:

- STIT 'between' Poisson hyperplane tessellations and Poisson-Voronoi tessellations
- Mixing properties for dimension $d = 2$ different from those ones for $d \geq 3$.