EXERCISES ON STABILITY CONDITIONS AND DONALDSON-THOMAS INVARIANTS

CIARAN MEACHAN

1. Stability Conditions

Exercise 1.1. Show that if $\sigma = (Z, \mathcal{P})$ is a stability condition on a triangulated category $\mathcal{D}$ then each subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ is abelian.

(Hint: Observe that $\mathcal{P}(> \phi)$ is a t-structure and deduce that $\mathcal{P}((\phi, \phi + 1])$ is abelian for any $\phi \in \mathbb{R}$.)

Solution. [Bri07, Lemma 3.4 & 5.2] or [Huy11, Corollary 2.8].

Exercise 1.2. Show that a stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a stability function on its heart $\mathcal{A}$ with the Harder-Narasimhan property.

(Hint: Use Exercise 1.1.)

Solution. [Bri07, Proposition 5.3].

Exercise 1.3. Let X be a smooth projective variety with $\dim(X) \geq 2$. Show that $\mathrm{Coh}(X)$ cannot be the heart of a numerical stability condition $\sigma = (Z, \mathcal{A})$.

(Hint: Consider the image of skyscraper sheaves $\mathcal{O}_x$ of points $x \in X$ under Z.)

Solution. [Tod09, Lemma 2,7] or [Huy11, Corollary 3.3]) when $\dim(X) = 2$.

Exercise 1.4. Let X be a d-dimensional smooth projective variety, and $\mathcal{B}_{B,\omega}$ a tilting of $\mathrm{Coh}(X)$ with respect to μ_ω-stability. Show that any non-zero $E \in \mathcal{B}_{B,\omega}$ satisfies the following:

(1) $\omega^{d-1}\mathrm{ch}_1^B(E) \geq 0$.

(2) If $\omega^{d-1}\mathrm{ch}_1^B(E) = 0$, then $\omega^{d-2}\mathrm{ch}_2^B(E) \geq 0$, $\mathrm{ch}_0^B(E) \leq 0$.

(3) If $\omega^{d-1}\mathrm{ch}_1^B(E) = \omega^{d-2}\mathrm{ch}_2^B(E) = \mathrm{ch}_0^B(E) = 0$, then $E \in \mathrm{Coh}(X)$ with $\dim \mathrm{Supp}(E) \leq d - 3$.

Deduce that $Z_{B,\omega}(\mathcal{B}_{B,\omega} \setminus \{0\}) \subset \mathbb{H}$ and $(Z_{B,\omega}, \mathcal{B}_{B,\omega}) \in \mathrm{Stab}(X)$ when $\dim(X) = 2$..

(Hint: Use the Hodge-Index theorem and the Bogomolov inequality.)

Solution. [BMT14, Lemma 3.2.1], [Bri08, Lemma 6.2] or [AB13, Corollary 2.1] when $\dim(X) = 2$. See [HL10, Section 3.4] for the Bogomolov inequality.

Exercise 1.5. Let C be a smooth projective curve C of genus $g(C) > 0$. Show that all skyscraper sheaves $\mathcal{O}_x$ of points $x \in C$, and all line bundles $\mathcal{L} \in \operatorname{Pic}(C)$ are stable with respect to a numerical stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}(C)$.

(Hint: Use the fact (or try to prove) that if a coherent sheaf E is included in a triangle $Y \to E \to X \to Y[1]$ with $\operatorname{Hom}^{\leq 0}(Y, X) = 0$ then $X, Y \in \operatorname{Coh}(C)$.)

Solution. [Mac07, Theorem 2.7] or [Huy11, Lemma 2.16].

Exercise 1.6. Show that the space of numerical stability conditions $\operatorname{Stab}(C)$ on $\mathcal{D}(C)$ of a smooth projective curve C of genus $g(C) > 0$ over $\mathbb{C}$ consists of exactly one $\widetilde{\operatorname{GL}}^+(2, \mathbb{R})$-orbit:

$$\operatorname{Stab}(C)/\widetilde{\operatorname{GL}}^+(2, \mathbb{R}) \simeq \{\mathrm{pt}\}.$$

Think about what happens when $g(C) = 0$, i.e. $C \simeq \mathbb{P}^1$?

(Hint: Observe the natural right action $\mathbb{C} \times \operatorname{GL}^+(2, \mathbb{R}) \to \mathbb{C}$; $(z, M) \mapsto M^{-1} \cdot z$ via the identification $\mathbb{C} \simeq \mathbb{R}^2$. This gives rise to a natural action

$$K(\mathcal{D})^* \times \operatorname{GL}^+(2, \mathbb{R}) \to K(\mathcal{D})^* \ ; \ (z, M) \mapsto M^{-1} \cdot z,$$

where $K(\mathcal{D})^* := \operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$, which can be lifted to an action on $\operatorname{Stab}(\mathcal{D})$ under the projection map $\pi : \operatorname{Stab}(\mathcal{D}) \to K(\mathcal{D})^*$; $(Z, \mathcal{A}) \to Z$ (after taking the universal cover and the connected component ith positive determinant).)

Solution. [Bri07, Section 9], [Mac07, Theorem 2.7] or [Huy11, Theorem 2.15].

Exercise 1.7. Let X be a smooth projective K3 surface. The minimal (and hence stable) objects in $\mathcal{B}_{B,\omega}$ are precisely the skyscraper sheaves $\mathcal{O}_x$ for all points $x \in X$ and shifts $F[1]$ of μ_ω-stable vector bundles F with $\mu_\omega(F) = B \cdot \omega$.

Solution. [Huy08, Proposition 2.2].

Exercise 1.8. Let X be a smooth projective surface and $\sigma = (Z, \mathcal{A})$ be a numerical stability condition on $\mathcal{D}(X)$ such that all skyscraper sheaves $\mathcal{O}_x$, $x \in X$, are σ-stable of phase 1. Show that the heart $\mathcal{A}$ of σ is of the form $\mathcal{B}_{B,\omega}$ for some $B, \omega \in \operatorname{NS}(X) \otimes \mathbb{R}$ with $\omega \in \operatorname{Amp}(X)$.

Solution. [Bri07, Proposition 10.3] or [Huy11, Theorem 3.2].

Exercise 1.9. Let X be a smooth projective threefold. For any $x \in X$, show that the skyscraper sheaf $\mathcal{O}_x$ is a minimal object in $\mathcal{A}_{B,\omega}$.

Solution. [MP13, Proposition 2.1].

Exercise 1.10. Let $\mathcal{M}_{B,\omega}$ be the class of all objects $E \in \mathcal{B}_{B,\omega}$ such that

(1) E is $\nu_{B,\omega}$-stable,

(2) $\nu_{B,\omega}(E) = 0$, and

(3) $\mathrm{Ext}^1_{\mathcal{M}}(\mathcal{O}_x, E) = 0$ for any skyscraper sheaf $\mathcal{O}_x$ of $x \in X$.

If $E \in \mathcal{M}_{B,\omega}$ then show that $E[1]$ is a minimal object of $\mathcal{A}_{B,\omega}$.

Solution. [MP13, Lemma 2.3].

Exercise 1.11. Let $0 \to E \to E' \to Q \to 0$ be a non splitting short exact sequence in $\mathcal{B}_{B,\omega}$ with $Q \in \mathrm{Coh}^0(X)$, $\mathrm{Hom}_{\mathcal{M}}(\mathcal{O}_x, E') = 0$ for any $x \in X$, and $\omega^2 \mathrm{ch}_1^B(E) \neq 0$. If E is $\nu_{B,\omega}$-stable then show that E' is $\nu_{B,\omega}$-stable.

Solution. [LM12, Proposition 3.5].

Exercise 1.12. Let $E \in \mathcal{B}_{B,\omega}$ be a $\nu_{B,\omega}$-semistable object with $\nu_{B,\omega}(E) < +\infty$. Then show that $H^{-1}_{\mathrm{Coh}(X)}(E)$ is a reflexive sheaf.

Solution. [LM12, Proposition 3.1].

Exercise 1.13. Let $E \in \mathcal{B}_{B,\omega}$ be a $\nu_{B,\omega}$-stable object with $\nu_{B,\omega}(E) = 0$. Then show that there exists $E' \in \mathcal{M}_{B,\omega}$ (that is $E'[1]$ is a minimal object in $\mathcal{A}_{B,\omega}$) such that

$$0 \to E \to E' \to Q \to 0$$

is a short exact sequence in $\mathcal{B}_{B,\omega}$ for some $Q \in \mathrm{Coh}^0(X)$. Then deduce the requirement of (weak or strong) Bogomolov-Gieseker type inequality only for objects in $\mathcal{M}_{B,\omega}$.

(Hint: Use the fact that $\mathcal{A}_{B,\omega}$ is Noetherian.)

Solution. [MP13, Proposition 2.9].

Exercise 1.14. Let X be a smooth projective variety of dimension n. Let $\sigma = (Z, \mathcal{P})$ be a locally finite numerical stability condition on $\mathcal{D}(X)$ such that all skyscraper sheaves $\mathcal{O}_x$ of $x \in X$ are σ-stable with phase one. Then show the following:

(1) if $E \in \mathcal{P}((0,1])$ then $H^i_{\mathrm{Coh}(X)}(E) = 0$ for $i \notin \{-n+1, -n+2, \ldots, 0\}$,

(2) if $E \in \mathcal{P}(1)$ is σ-stable then $E \cong \mathcal{O}_x$ for some $x \in X$, or E is a complex such that $H^i_{\mathrm{Coh}(X)}(E) = 0$ for $i \notin \{-n+1, -n+2, \ldots, -1\}$, and

(3) if $E \in \mathrm{Coh}(X)$ then $E \in \mathcal{P}((-n+1, 1])$.

Solution. [Bri08, Lemma 10.1] for the $n = 2$ case.

Exercise 1.15. If $\sigma = (Z, \mathcal{P})$ is a locally finite numerical stability condition on $D^b(X)$, then observe that σ-stable objects in $\mathcal{P}(1)$ are minimal objects in the heart $\mathcal{P}((0,1])$. As a result of the above exercise, obtain that if $E \in \mathcal{P}(1)$ is stable then we have the following:

(1) when $\dim X = 1$, $E \cong \mathcal{O}_x$ for some $x \in X$;

(2) when $\dim X = 2$, $E \cong \mathcal{O}_x$ for some $x \in X$ or $E \cong F_{-1}[1]$ for some locally free sheaf F_{-1}; and

(3) when $\dim X = 3$, $E \cong \mathcal{O}_x$ for some $x \in X$ or E is quasi isomorphic to a complex $0 \to F_{-2} \to F_{-1} \to 0$ of locally free sheaves, and so $H^{-2}_{\mathrm{Coh}(X)}(E)$ is a reflexive sheaf.

Solution. Compare this with Exercise 1.12.

2. Donaldson-Thomas Invariants

Exercise 2.1. Let X be a smooth projective variety over $\mathbb{C}$ and H an ample divisor on X. Recall that the Hilbert polynomial of a coherent sheaf E of dimension d is given by

$$\chi(E, m) := \chi(E \otimes \mathcal{O}(mH)) := \sum_{i=0}^{d} (-1)^i h^i(X, E \otimes \mathcal{O}(mH)) \quad \text{for any } m \in \mathbb{Z}.$$

Use Riemann-Roch to show that this can be written as

$$\chi(E, m) = \alpha_d m^d + \alpha_{d-1} m^{d-1} + \cdots \quad \text{for } \alpha_i \in \mathbb{Q}.$$

(Hint: Use the fact that a line bundle L has Chern character $\mathrm{ch}(L) = \exp(c_1(L))$.)

Solution. Direct computation or see [HL10, Lemma 1.2.1] for more details.

Exercise 2.2. Recall the theorem of Joyce-Song for a smooth projective Calabi-Yau threefolds: it says that for all $p \in \mathcal{M}^s_H(v)$, there exists an analytic open set $\mathcal{U} \subset \mathcal{M}^s_H(v)$ containing p such that $\mathcal{U} = \{df = 0\}$ for some holomorphic function $f : V \to \mathbb{C}$ on a complex manifold V. Then Behrend's constructible function is defined as

$$\nu : \mathcal{M}^s_H(v) \to \mathbb{Z} \, ; \, p \mapsto (-1)^{\dim V}(1 - \chi(M_p(f)))$$

where $M_p(f) := \{x \in V : \|x - p\| < \delta, \; f(x) = f(p) + \epsilon, \; 0 < \epsilon \ll \delta \ll 1\}$ is the Milnor fibre of f at p. Show that if $\mathcal{M}^s_S(v) := \mathrm{Spec}(\mathbb{C}[t]/t^m)$ then $\nu \equiv m$.

(Hint: Find a complex manifold V with a holomorphic function f on it such that $\mathrm{Spec}(\mathbb{C}[t]/t^m) = \{df = 0\}$ and compute the Euler characteristic of the fibres.)

Solution. Set $V := \mathrm{Spec}(\mathbb{C}[t])$ and $f := t^{m+1}$. Then $\dim V = 1$ and there are $m+1$ points in the fibre which implies $\chi(M_p(f)) = m+1$. In other words, we have $\nu := (-1)^{\dim V}(1 - \chi(M_p(f))) = (-1)^1(1 - (m+1)) = m$.

Exercise 2.3. Recall that the "honest" curve count is defined as

$$I'_\beta(X) := \frac{I_\beta(X)}{M(-q)^{\chi(X)}}$$

where $I_\beta(X) := \sum_n I_{n,\beta} q^n$, $I_{n,\beta} := \mathrm{DT}_H(1,0,-\beta,-n)$ and

$$M(q) := \prod_{m\geq 1}(1-q^m)^{-m}$$

is the MacMahon function. If $\mathbb{P}^1 \simeq C \hookrightarrow X$ is a contractible $(-1,-1)$-curve in a Calabi-Yau threefold then

$$\sum_{m\geq 0} I_{m[C]}(X)t^m = M(-q)^{\chi(X)} \cdot \prod_{k\geq 1}(1-(-q)^k t)^k. \tag{1}$$

Use this to show that

$$I'_{[C]}(X) = q - 2q^2 + 3q^3 - \cdots = \frac{q}{(1+q)^2} = \frac{1/q}{(1+(1/q))^2}.$$

Solution. Set $m = 1$ and compute the coefficient of t. Indeed, Equation (1) says that

$$I'_0(X) + I'_{[C]}(X)t + I'_{[2C]}(X)t^2 + \cdots = (1+qt)(1-q^2t)^2(1+q^3t)^3\cdots$$

and so $I'_{[C]}(X)$ is equal to the coefficient of t on the right hand side.

Exercise 2.4. Recall from the lectures that we have

$$\begin{aligned}\sum_\beta I_\beta(X)t^\beta &= \prod_{n\geq 0}\exp((-1)^{n-1}nN_{n,0}q^n)\cdot(\sum_\beta P_\beta(X)t^\beta)\\ &= \prod_{\substack{n\geq 0\\ \beta\geq 0}}\exp((-1)^{n-1}nN_{n,\beta}q^n t^\beta)\cdot(\sum_\beta L_\beta(X)t^\beta)\end{aligned}$$

where $L_\beta(X) := \sum_{n\in\mathbb{Z}} L_{n,\beta}q^n$ is a polynomial of $q^\pm$ invariant under $q \leftrightarrow 1/q$ and $N_{n,\beta} \in \mathbb{Q}$, $L_{n,\beta} \in \mathbb{Z}$ are such that $N_{n,\beta} = N_{-n,\beta} = N_{n+H\cdot\beta,\beta}$, $L_{n,\beta} = L_{-n,\beta}$ and $L_{n,\beta} = 0$ for $|n| \gg 0$. Use this result to show that

$$I'_\beta(X) = \frac{I_\beta(X)}{M(-q)^{\chi(X)}} = P_\beta(X).$$

Furthermore, use the periodicity of $N_{n,\beta}$ to show that $I'_\beta(X)$ is a rational function of q, invariant under $q \leftrightarrow 1/q$.

Solution. First observe that $\prod_{n\geq 0}\exp((-1)^{n-1}nN_{n,0}q^n)$ is independent of β and so if we set $\beta = 0$ then the first equality gives

$$I_0(X) = \prod_{n\geq 0}\exp((-1)^{n-1}nN_{n,0}q^n)\cdot P_0(X).$$

Next, we see that $P_0(X) = 1$ by definition of stable pairs and $I_0(X) = M(-q)^{\chi}(X)$ by Lecture 1. Thus, we have $M(-q)^{\chi}(X) = \prod_{n\geq 0} \exp((-1)^{n-1} n N_{n,0} q^n)$ and hence

$$I'_\beta(X) := \frac{I_\beta(X)}{M(-q)^{\chi}(X)} = \frac{\prod_{n\geq 0} \exp((-1)^{n-1} n N_{n,0} q^n) \cdot P_\beta(X)}{M(-q)^{\chi}(X)} = P_\beta(X).$$

The rationality of $I'_\beta(X)$ is proved in [Tod10, Lemma 4.6].

References

[AB13] Daniele Arcara and Aaron Bertram. Bridgeland-stable moduli spaces for K-trivial surfaces. *J. Eur. Math. Soc. (JEMS)*, 15(1):1–38, 2013. With an appendix by Max Lieblich.

[BMT14] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.*, 23(1):117–163, 2014.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[Huy08] Daniel Huybrechts. Derived and abelian equivalence of $K3$ surfaces. *J. Algebraic Geom.*, 17(2):375–400, 2008.

[Huy11] Daniel Huybrechts. Introduction to stability conditions. *Arxiv preprint arXiv:1111.1745v2*, 2011.

[LM12] Jason Lo and Yogesh More. Some examples of tilt-stable objects on threefolds. *Arxiv preprint arXiv:1209.2749v1*, 2012.

[Mac07] Emanuele Macrì. Stability conditions on curves. *Math. Res. Lett.*, 14(4):657–672, 2007.

[MP13] Antony Maciocia and Dulip Piyaratne. Fourier-mukai transforms and bridgeland stability conditions on abelian threefolds. *Arxiv preprint arXiv:1304.3887v3*, 2013.

[Tod09] Yukinobu Toda. Limit stable objects on Calabi-Yau 3-folds. *Duke Math. J.*, 149(1):157–208, 2009.

[Tod10] Yukinobu Toda. Generating functions of stable pair invariants via wall-crossings in derived categories. In *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*, volume 59 of *Adv. Stud. Pure Math.*, pages 389–434. Math. Soc. Japan, Tokyo, 2010.

School of Mathematics, University of Edinburgh, Scotland

E-mail address: `ciaran.meachan@ed.ac.uk`