Statistical methods for dynamical stochastic models - DYNSTOCH 2016, University of Rennes, June 8th to June 10th

Quasi likelihood analysis and limit order book modeling

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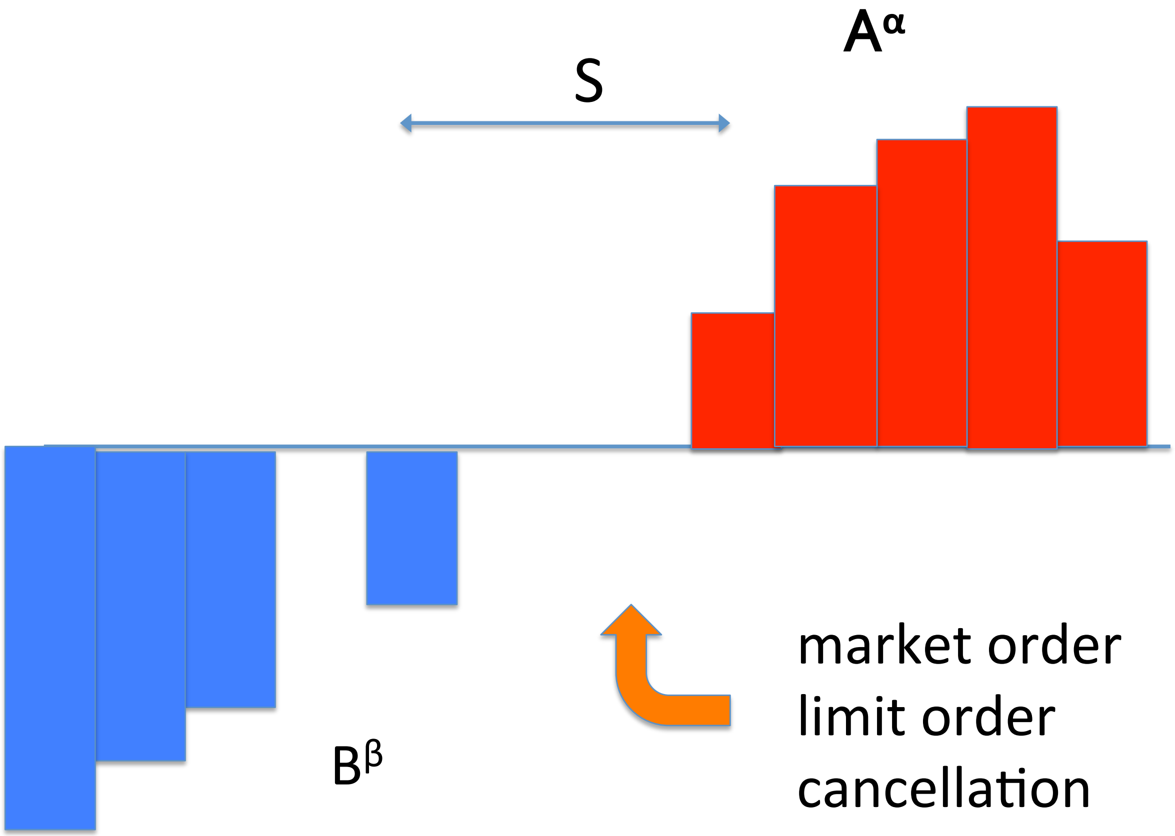
A Limit Order Book modeling

With Ioane Muni Toke

Limit Order Book modeling:



Limit Order Book modeling

- **LOB is described by the multi-dimensional stochastic process**

$$\mathbb{X} = ((A^{\alpha})_{\alpha=1,\dots,k_A}, (B^{\beta})_{\beta=1,\dots,k_B})$$

where

  - A_t^{α}: **total number of limit orders available at price (tick)** p_A^{α} **on the ask side at time** t
  - B_t^{β}: **total number of limit orders available at price (tick)** p_B^{β} **on the bid side at time** t

- **The state space of** $\mathbb{X}$ **is absolutely or relatively set:**
  - **the price** p_A^{α} **is at the relative** α**-th limit order from the best quote on the same/opposite side,**
    or p_A^{α} **is the absolute price**

- **The random evolution of $\mathbb{X}$ is determined by the processes**
  - **M^A counting number of arrivals of market orders on the ask side,**
  - **M^B of market orders on the bid side,**
  - **L^α of limit orders at level α on the ask side,**
  - **L^β of limit orders at level β on the bid side,**
  - **C^α of cancellation at level α on the ask side, and**
  - **C^β of cancellation at level β on the bid side.**
- **The multivariate counting process N^n consists of these counting processes. Here prices can be recognized as a function of $\mathbb{X}$.**
- **For modeling of C^α and C^β, we may treat $g^n(t, \theta)$ proportional to A^α and B^β, respectively, or more complicated mechanism.**

Counting arrivals of limit orders

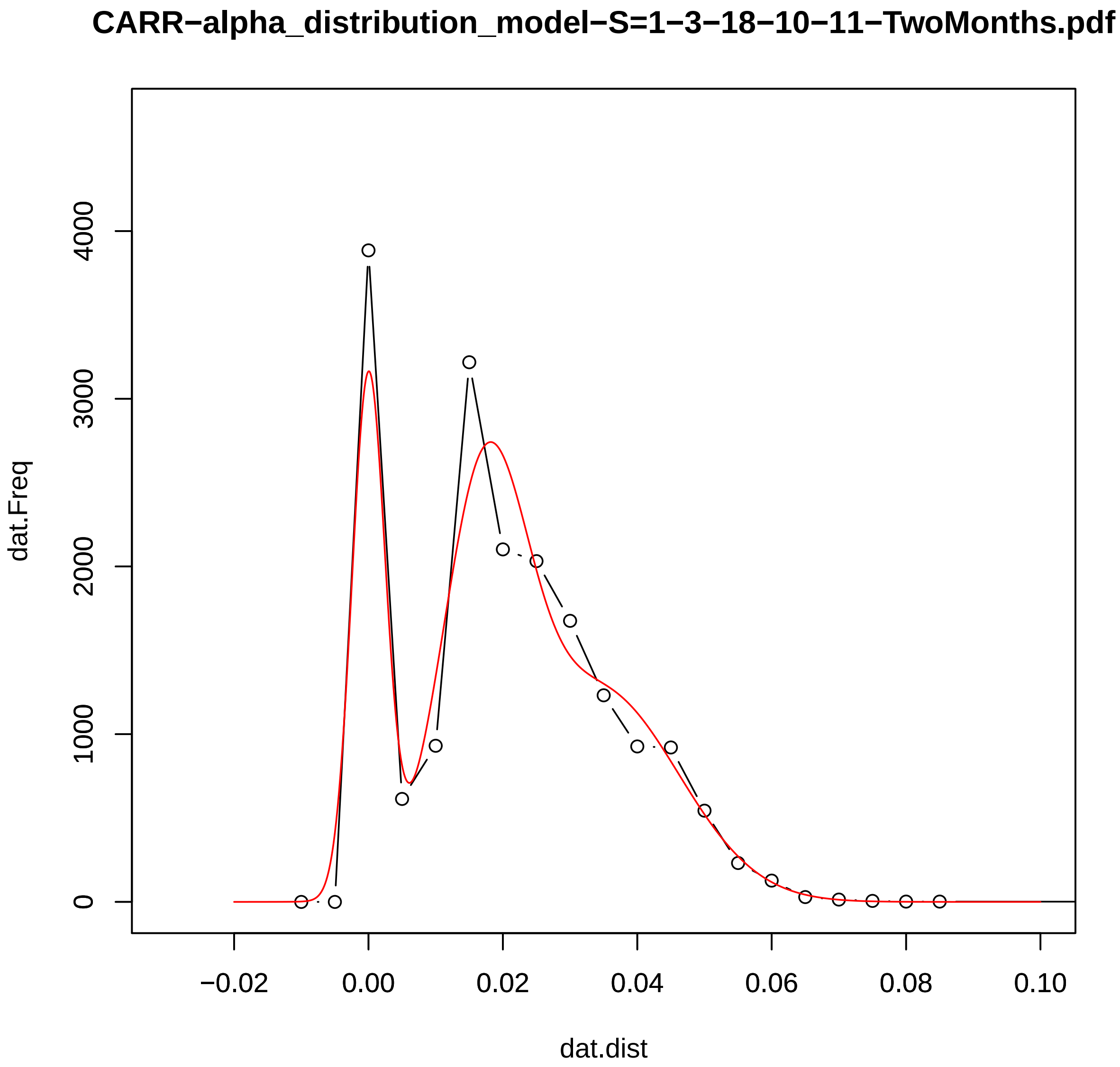


Figure 1: CARR: S-conditional alpha-distribution model (red)

Limit order intensities modeling

- **Modiling limit order intensities λ_α^{LA}**
  - **Discover covariates from the data and give a functional representation of λ_α^{LA}.**
  - **A relatively simple dependency has been found:**

$$\lambda_\alpha^{LA} = \lambda_\alpha^{LA}(S_t)$$

Limit order intensity model (Muni Toke and Y)

- Intensity model (spot form) is proposed as

$$\lambda_\alpha^{LA}(S) = \sum_{i=1}^{3} \Lambda_i(S)\phi(\alpha\delta; \mu_i(S), \mathrm{sd}_i(S)^2) \qquad (\alpha \in \mathbb{R})$$

  – Λ_i are positive functions of S

$$\Lambda_i(S) = \exp(\beta(S))\pi_i(S). \tag{1}$$

  – A model $\beta(s) = \sum_{j=0}^{2} \beta_j s^j$.

$$\mu_i(s) = \sum_{j=0}^{2} \mu_{i,j} s^j \qquad \mathrm{sd}_i(s) = \sum_{j=0}^{2} \sigma_{i,j} s^j$$

$$\pi_i(s) = \frac{\exp\left(\pi_{i,0} + \pi_{i,1}s + \pi_{i,2}s^2\right)}{\sum_{j=1}^{3} \exp\left(\pi_{j,0} + \pi_{j,1}s + \pi_{j,2}s^2\right)}$$

$$(\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0)$$

where δ is the tick size, $\mu_{i,j}$, $\sigma_{i,j}$ and $\pi_{i,j}$ are constants depending on the asset and the environment in the sampling period.

Limit order intensity model (by model)

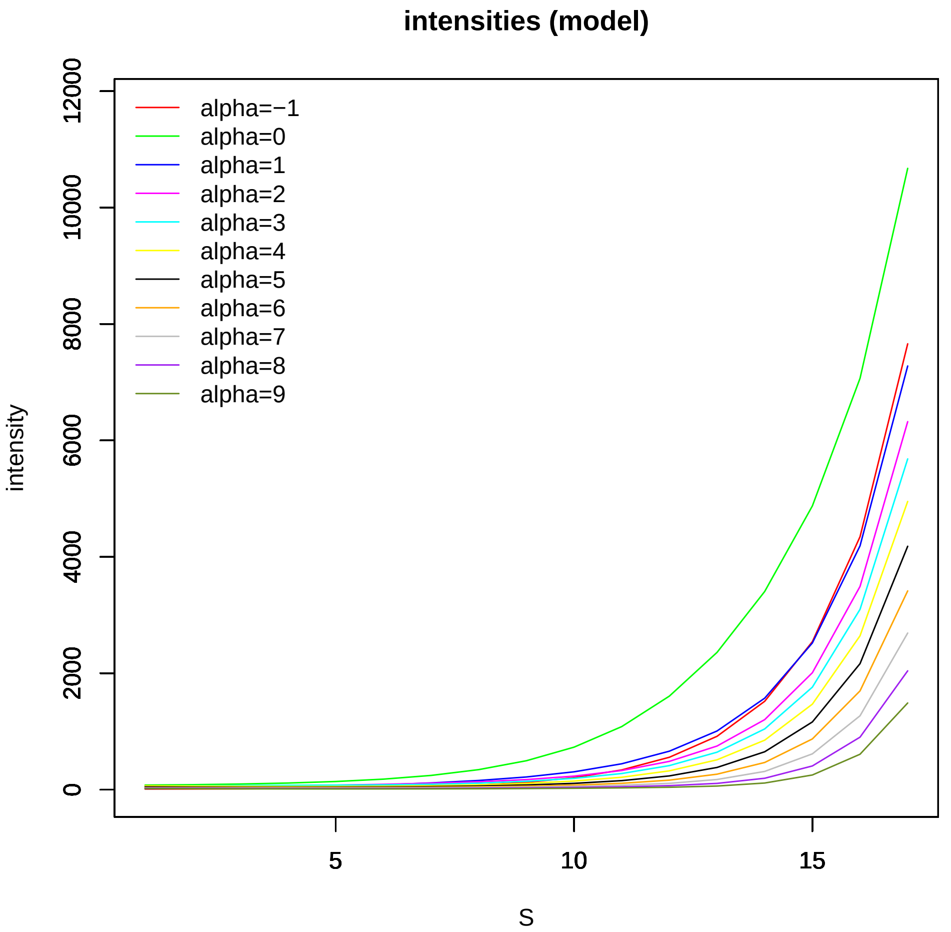


Figure 2: CARR: S and intensities for given $\boldsymbol{\alpha}$ by intensity model

Fit model to LOB arrival numbers data

- **Fit the model to the counting data of the numbers of limit orders for various spreads in a fixed time interval.**
- **Remark. The fitted values are not intensities but the expected numbers of limit orders in the time interval.**

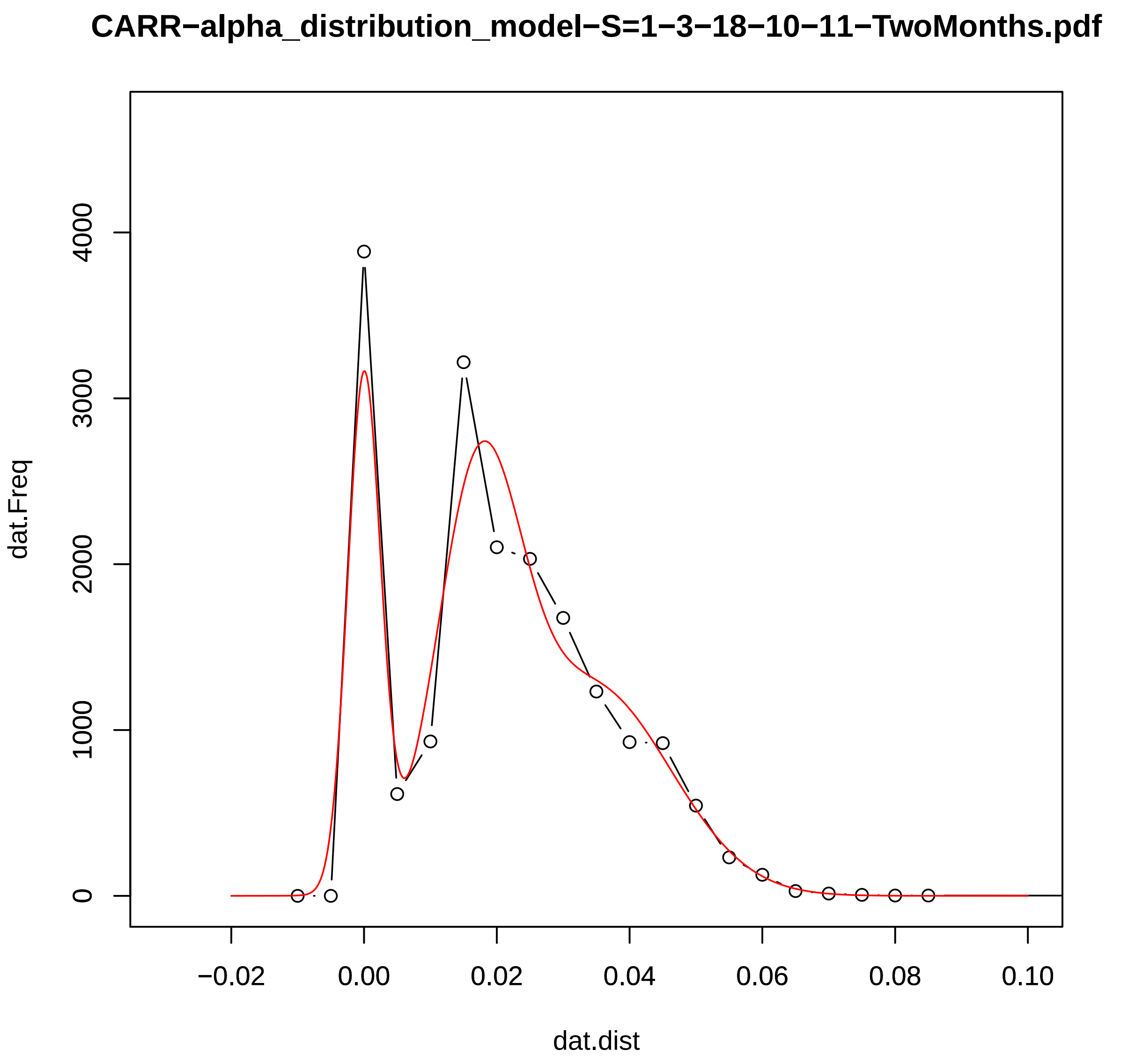


Figure 3: CARR: S-conditional alpha-distribution model (red)

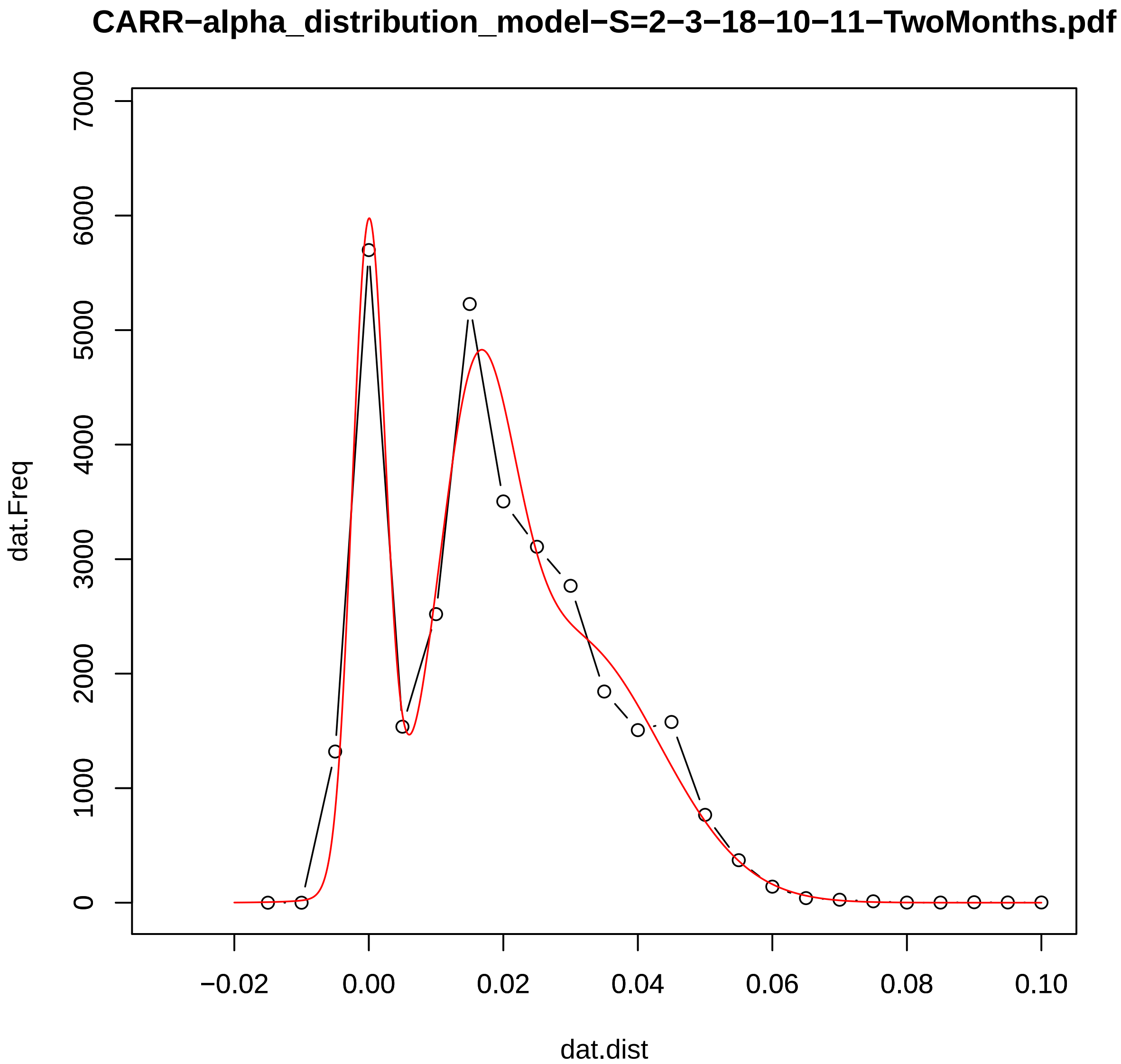


Figure 4: CARR: S-conditional alpha-distribution model (red)

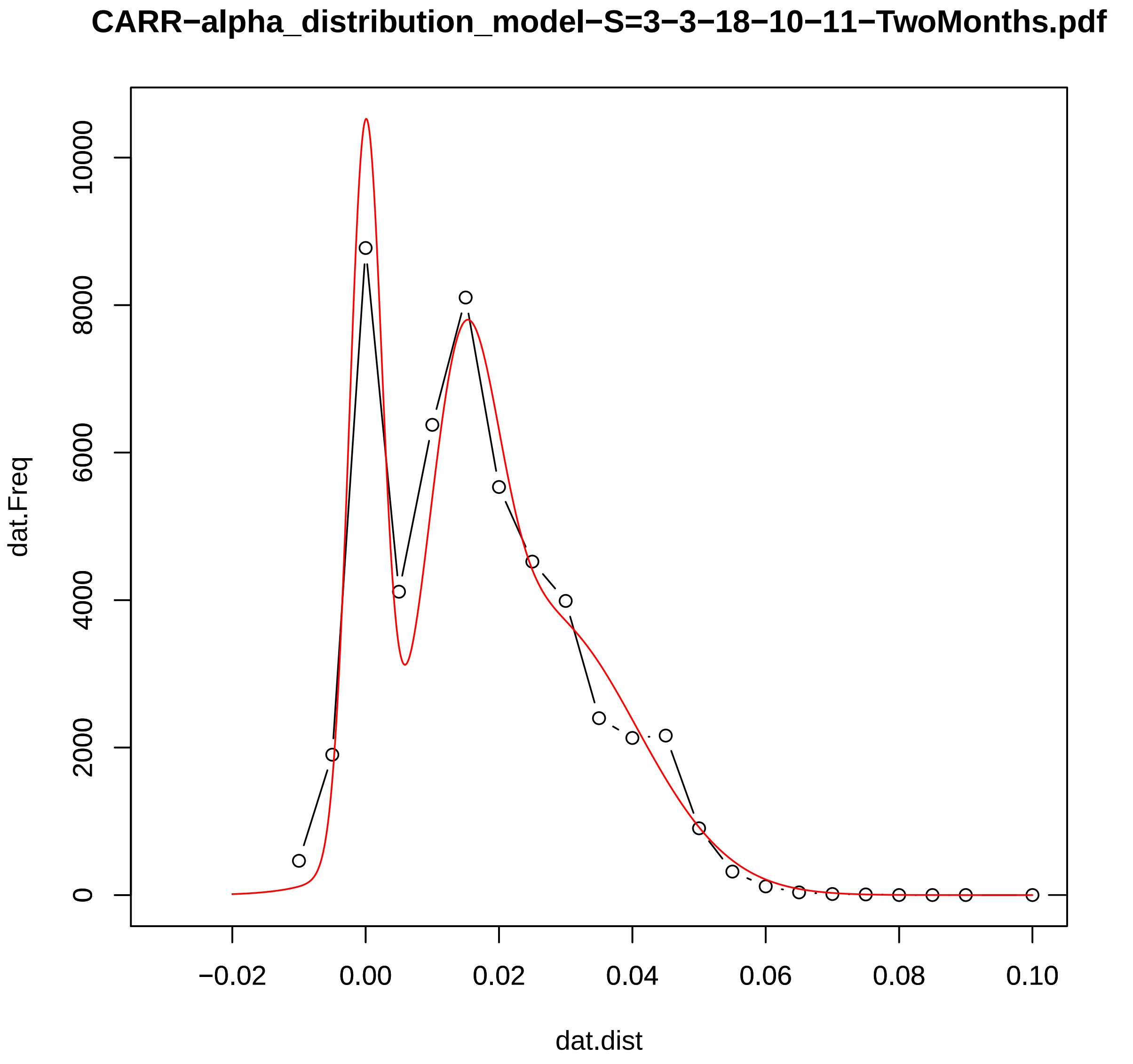


Figure 5: CARR: S-conditional alpha-distribution model (red)

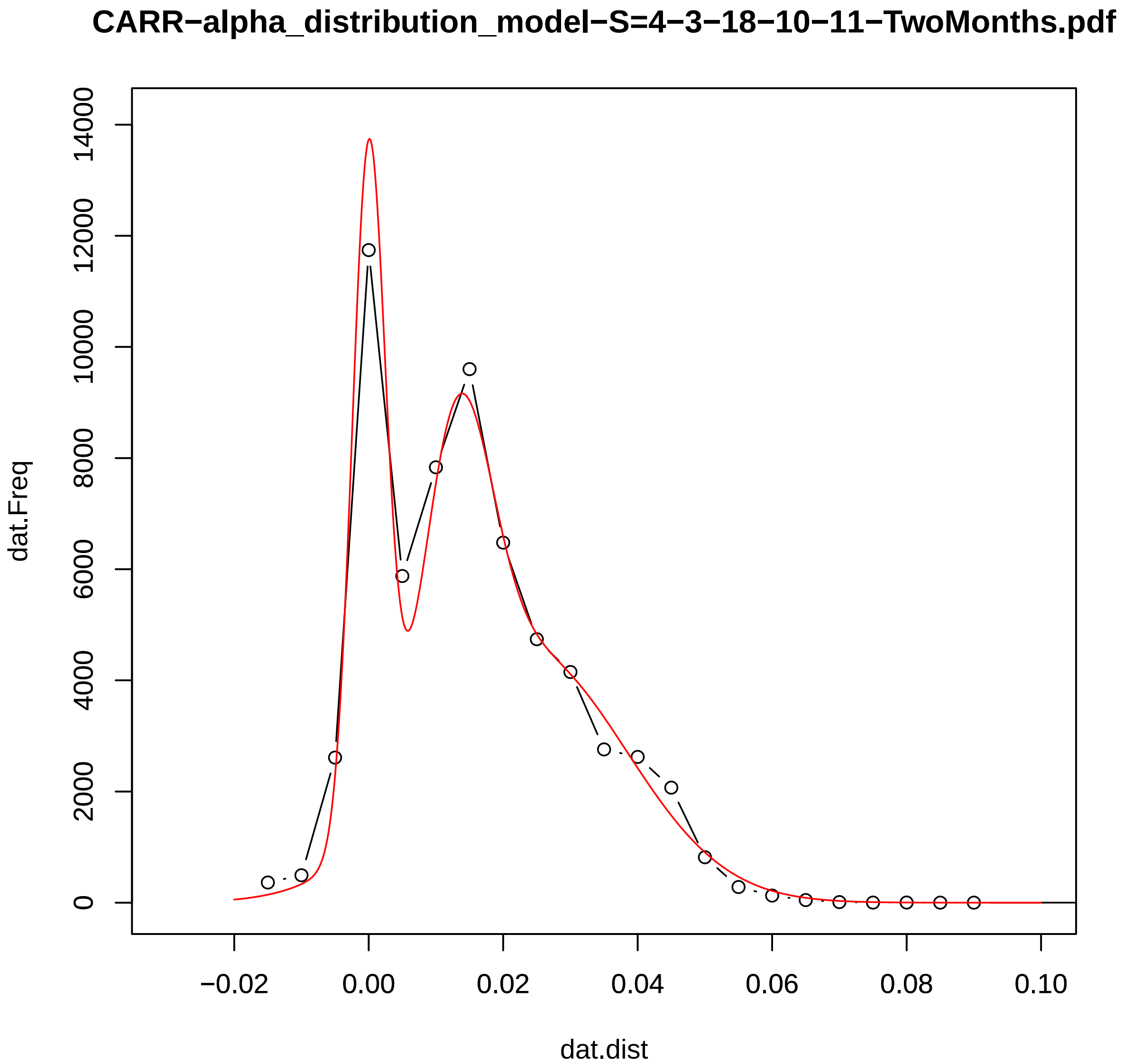


Figure 6: CARR: S-conditional alpha-distribution model (red)

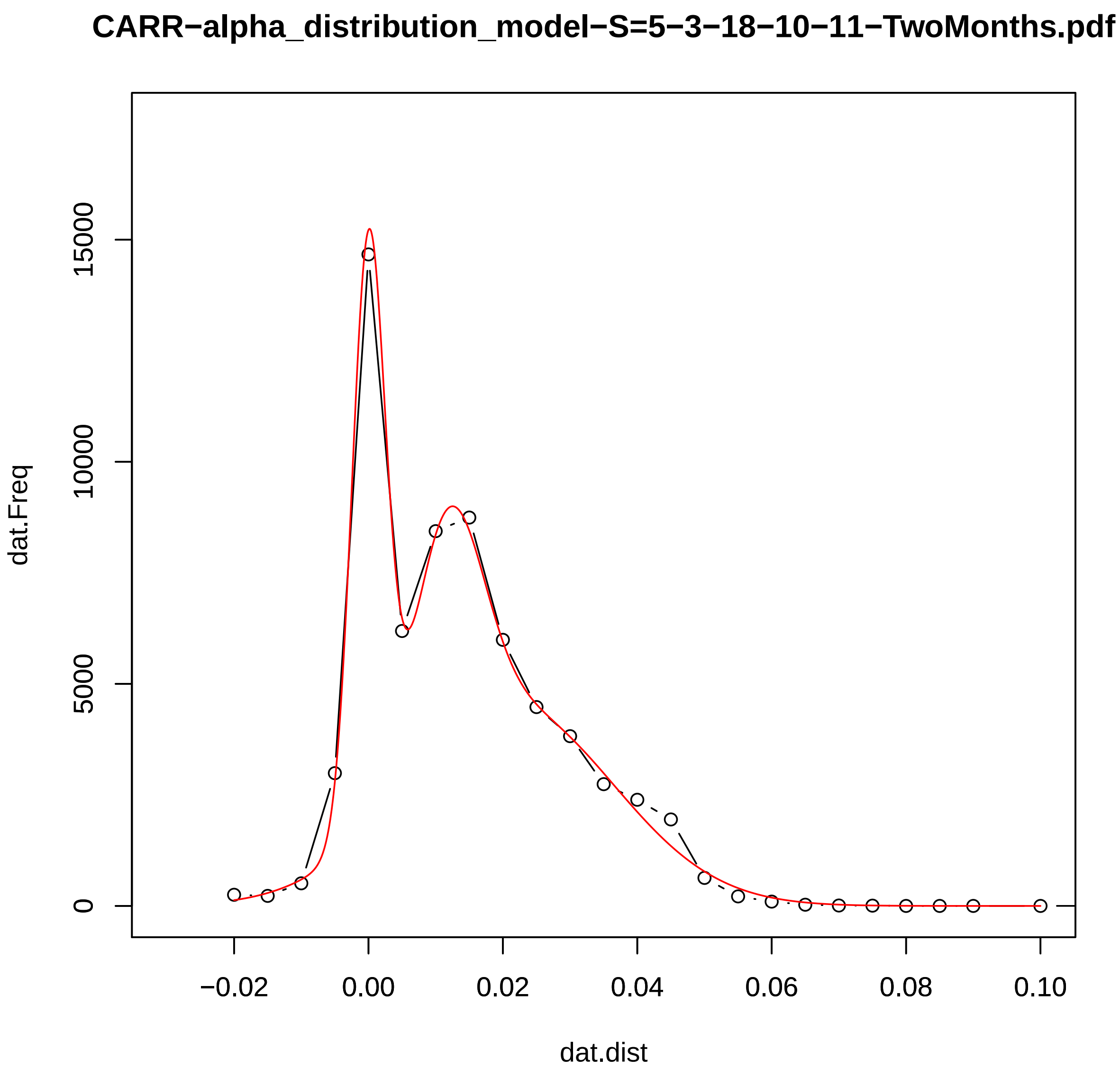


Figure 7: CARR: S-conditional alpha-distribution model (red)

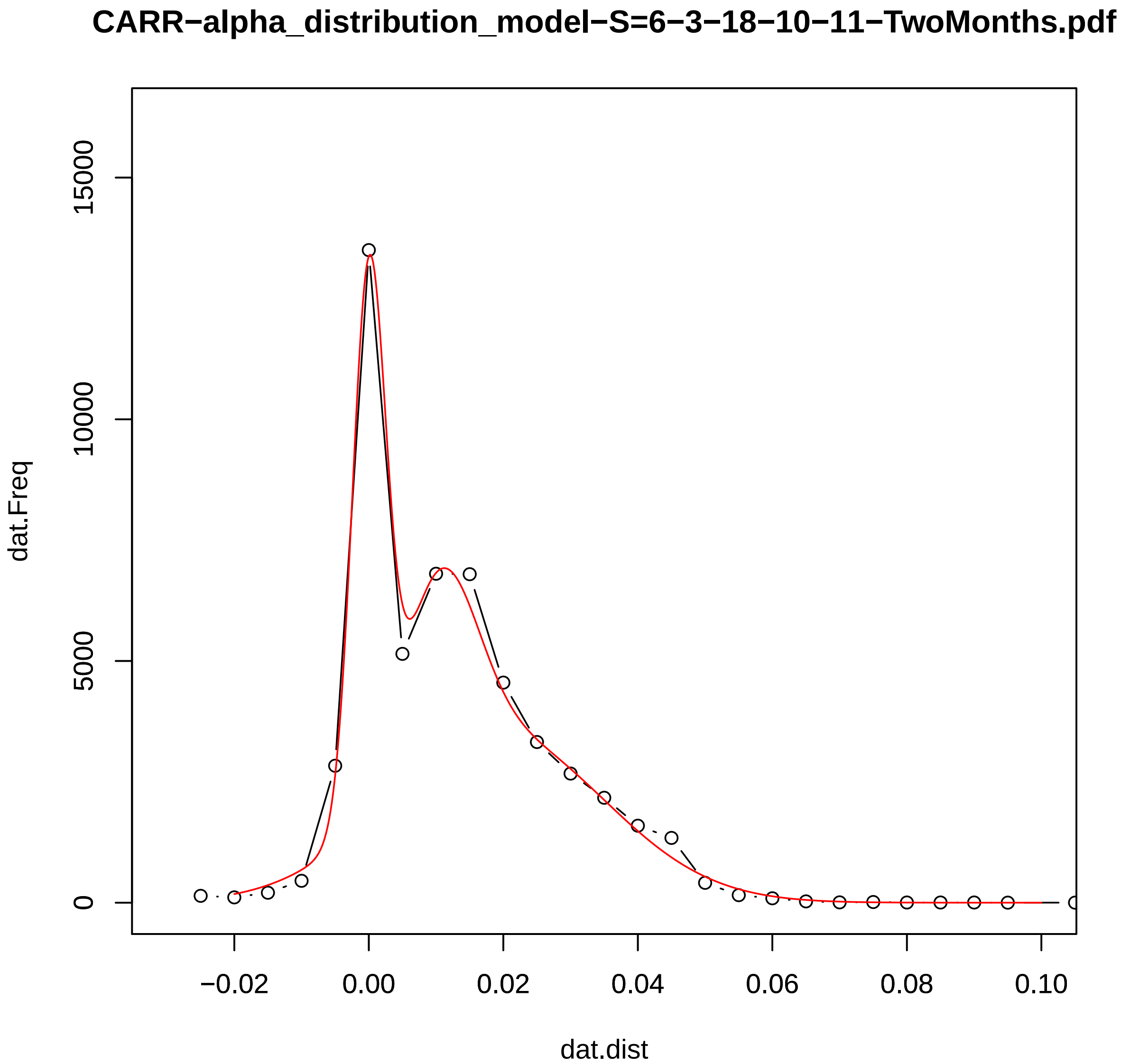


Figure 8: CARR: S-conditional alpha-distribution model (red)

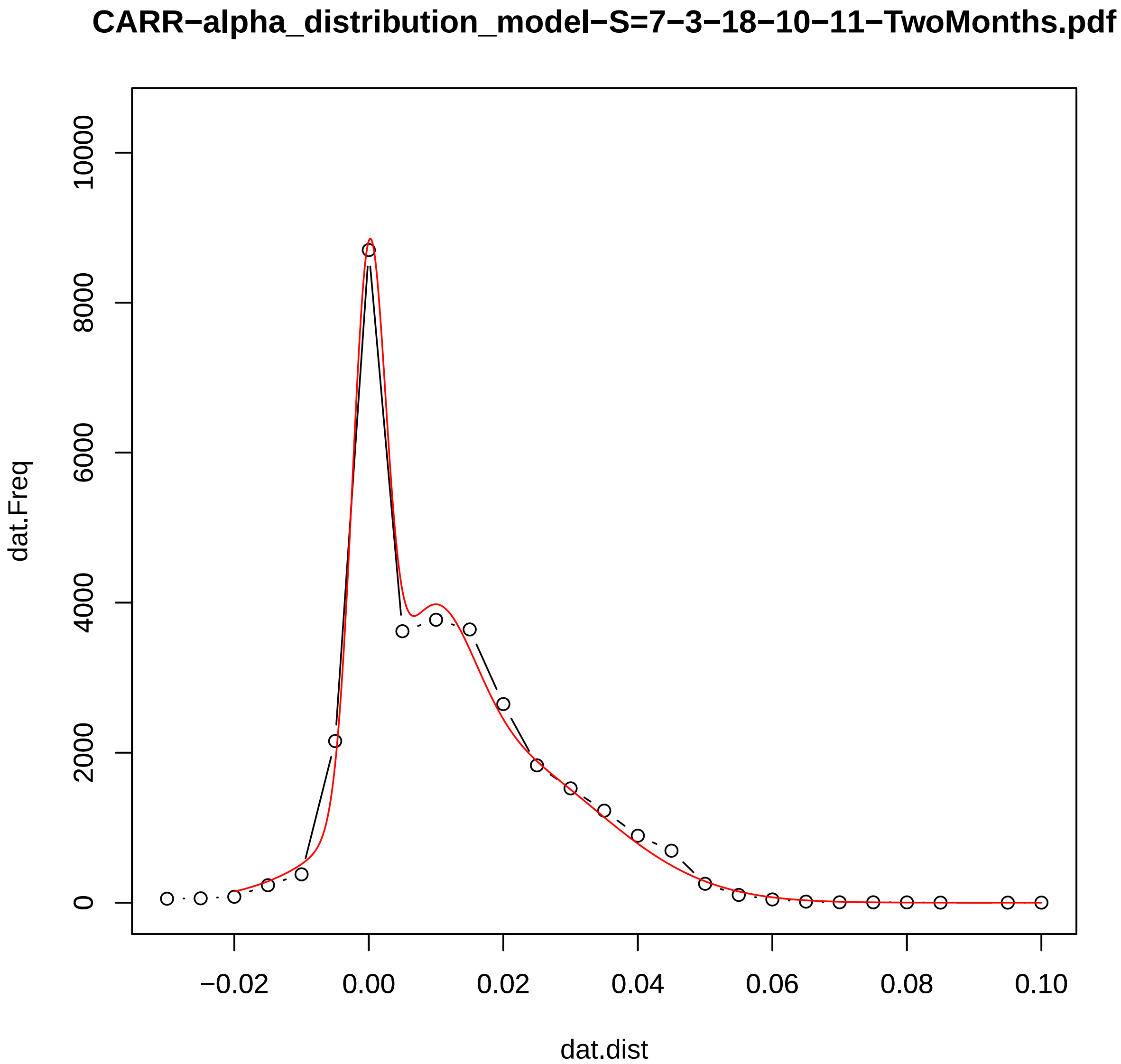


Figure 9: CARR: S-conditional alpha-distribution model (red)

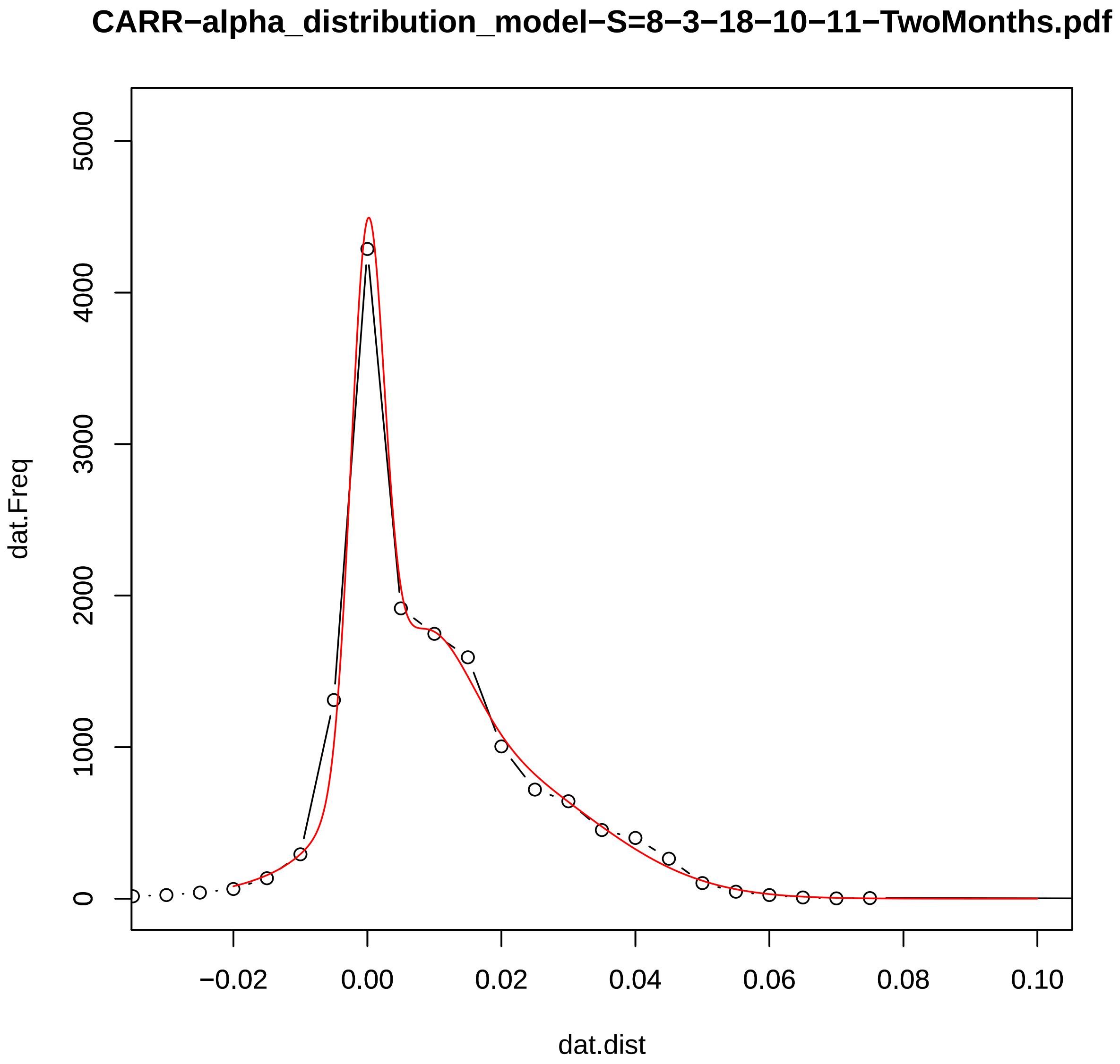


Figure 10: CARR: S-conditional alpha-distribution model (red)

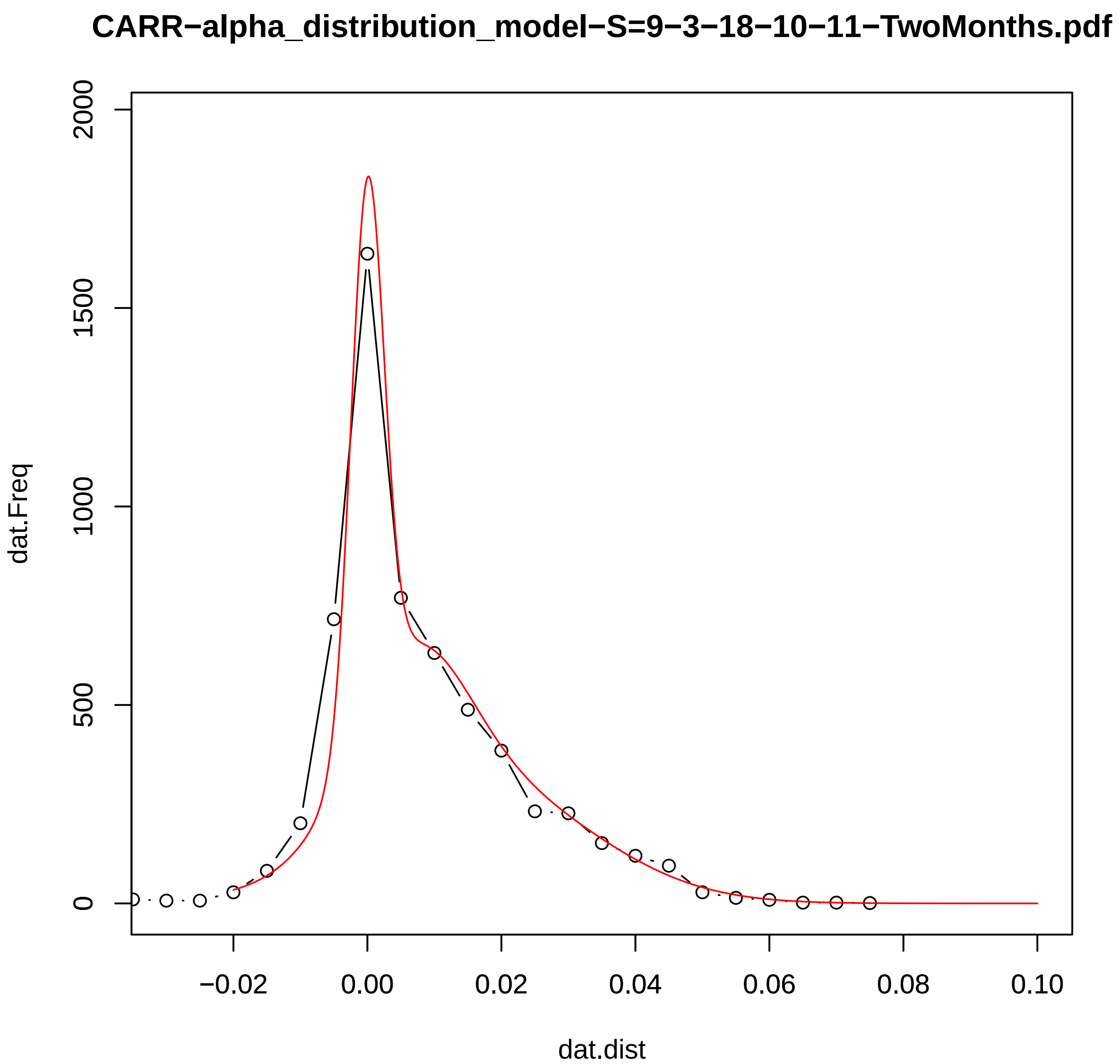


Figure 11: CARR: S-conditional alpha-distribution model (red)

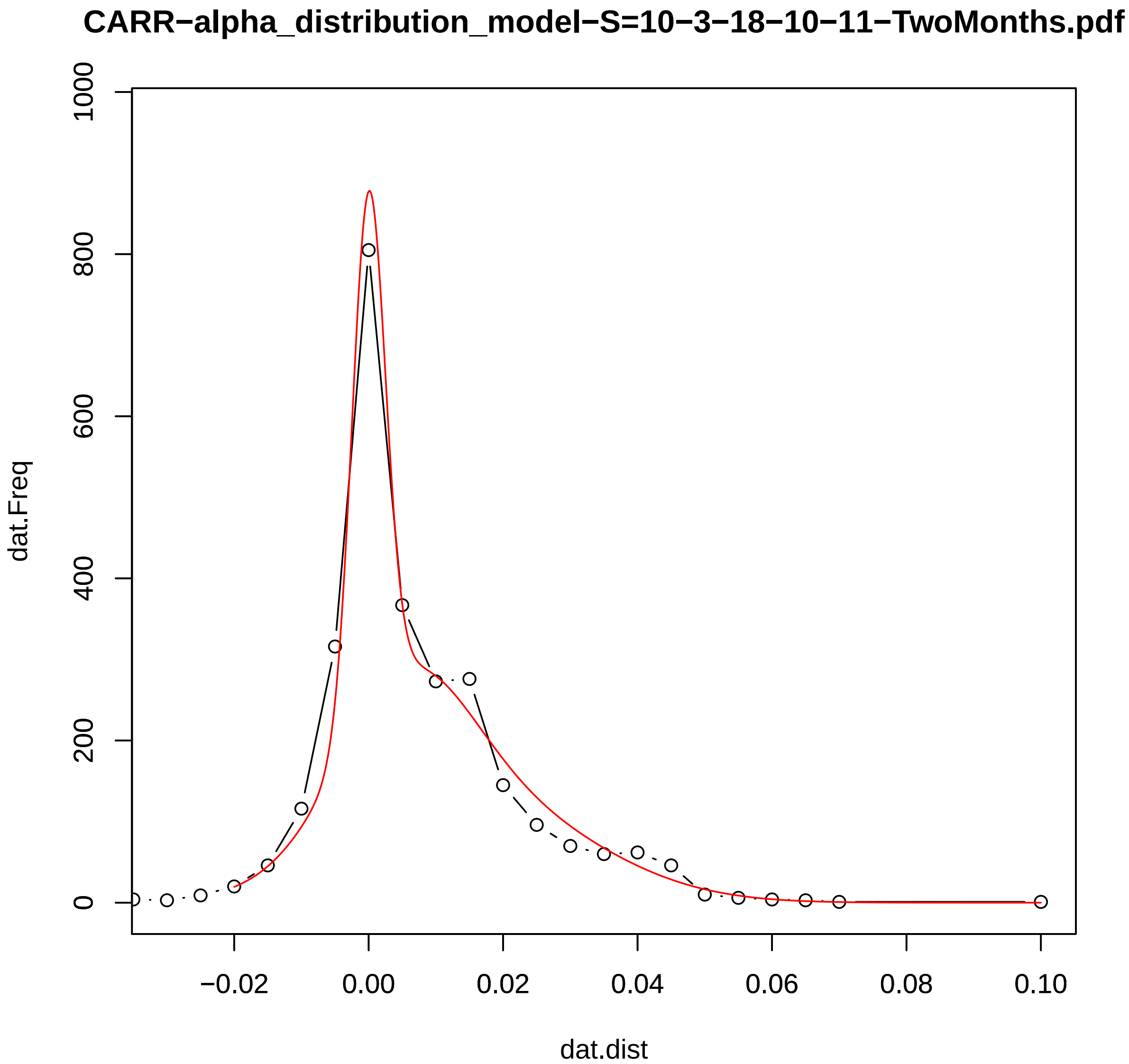


Figure 12: CARR: S-conditional alpha-distribution model (red)

Limit order book modeling

- **This analysis shows possibility of regression model with covariate processes.**

Ultra high frequency data and modeling by point processes

High frequency financial data

hpq_dis_12_2010 - メモ帳

ファイル(F) 編集(E) 書式(O) 表示(V) ヘルプ(H)

HPQ	12/01/2010	N	15:59:54	42.5800	700	F	0	0
HPQ	12/01/2010	N	15:59:54	42.5800	100	@	0	0
HPQ	12/01/2010	N	15:59:55	42.5800	100	@	0	0
HPQ	12/01/2010	N	15:59:55	42.5700	200	@	0	0
HPQ	12/01/2010	N	15:59:55	42.5800	394	@	0	0
HPQ	12/01/2010	N	15:59:55	42.5700	100	@	0	0
HPQ	12/01/2010	N	15:59:55	42.5800	100	@	0	0
HPQ	12/01/2010	N	15:59:57	42.5800	180	@	0	0
HPQ	12/01/2010	N	15:59:59	42.5800	6300	@	0	0
HPQ	12/01/2010	N	15:59:59	42.5800	100	@	0	0
HPQ	12/01/2010	N	15:59:59	42.5700	1000	@	0	0
HPQ	12/01/2010	N	15:59:59	42.5700	6200	@	0	0
HPQ	12/01/2010	N	15:59:59	42.5700	100	@	0	0
HPQ	12/01/2010	N	16:00:07	42.5700	1044254	6	0	0
DIS	12/01/2010	N	9:30:12	37.2300	182284	O	0	0
DIS	12/01/2010	N	9:30:12	37.2100	200	@	0	0
DIS	12/01/2010	N	9:30:12	37.2100	100	@	0	0
DIS	12/01/2010	N	9:30:12	37.2100	100	@	0	0
DIS	12/01/2010	N	9:30:12	37.2100	202	@	0	0
DIS	12/01/2010	N	9:30:12	37.2200	400	F	0	0
DIS	12/01/2010	N	9:30:13	37.1900	300	@	0	0
DIS	12/01/2010	N	9:30:15	37.2300	288	@	0	0
DIS	12/01/2010	N	9:30:15	37.2400	200	F	0	0
DIS	12/01/2010	N	9:30:15	37.2500	712	F	0	0
DIS	12/01/2010	N	9:30:15	37.2500	108	@	0	0
DIS	12/01/2010	N	9:30:15	37.2000	100	@	0	0
DIS	12/01/2010	N	9:30:16	37.2500	1584	@	0	0

Phenomena we want to model

- Epps effect (1979)
  A natural correlation estimator has a bias in high frequent observations
  - non-synchronicity of the observations
  - microstructure Observations are $X_{t_j} + \epsilon_j$?
    No BM in ultra high frequency sampling
- lead-lag
- relativity of prices — In Limit order Book,
  "Price" is a functional of the state of LOB.
- Dependency on covariates
- To incorporate these effects, we will consider
  a point process regression model.

Modeling high frequency data by point processes

- **Multivariate point process**
  - **Hewlett (2006)**
    the clustered arrivals of buy and sell trades using Hawkes processes
  - **Large (2007)**
    Extension by using a finer description of orders
  - **Bowsher (2007)**
    Generalized Hawkes model
  - **E. Bacry et al. (2013)**
    Price as "upward − downward counting processes"
  - **Chen and Hall (2013)**
    the intraday trading times of a common stock traded on the Australian Stock Exchange, the ANZ stock.

Modeling high frequency data by point processes

- **Limit order book**
  - **R. Cont, Stoikov and Talreja (2010)**
  - **Abergel and Jedidi (2013)**
  - **Smith, Farmer, Gillemot and Krishnamurthy (2003)**
  - **Muni Toke and Pomponio (2011)**

Point process regression model

Ogihara and Yoshida, arXiv 2015

Point process regression model

- The **d**-dimensional point process $N^n = (N^{n,\alpha})_{\alpha \in \mathcal{I}}$ on $I = [T_0, T_1]$, $\mathcal{I} = \{1, ..., \mathbf{d}\}$, is assumed to have

an intensity process $n\lambda^n(t, \theta)$ defined by

$$\lambda^n(t, \theta) = g^n(t, \theta) + \int_{\hat{T}_0}^{t-} K^n(t, s, \theta) dX_s^n,$$

where θ is a parameter and $\hat{T}_0 < T_0 < T_1$.

- Examples.

$$\lambda^n(t, \theta) = \lambda^\infty(t, \theta) = g(V_t, \theta)$$

$$\lambda^n(t, \theta) = \lambda^\infty(t, \theta) = g(t, \gamma) + \int_0^t e^{-b(t-s)} A V_s ds$$

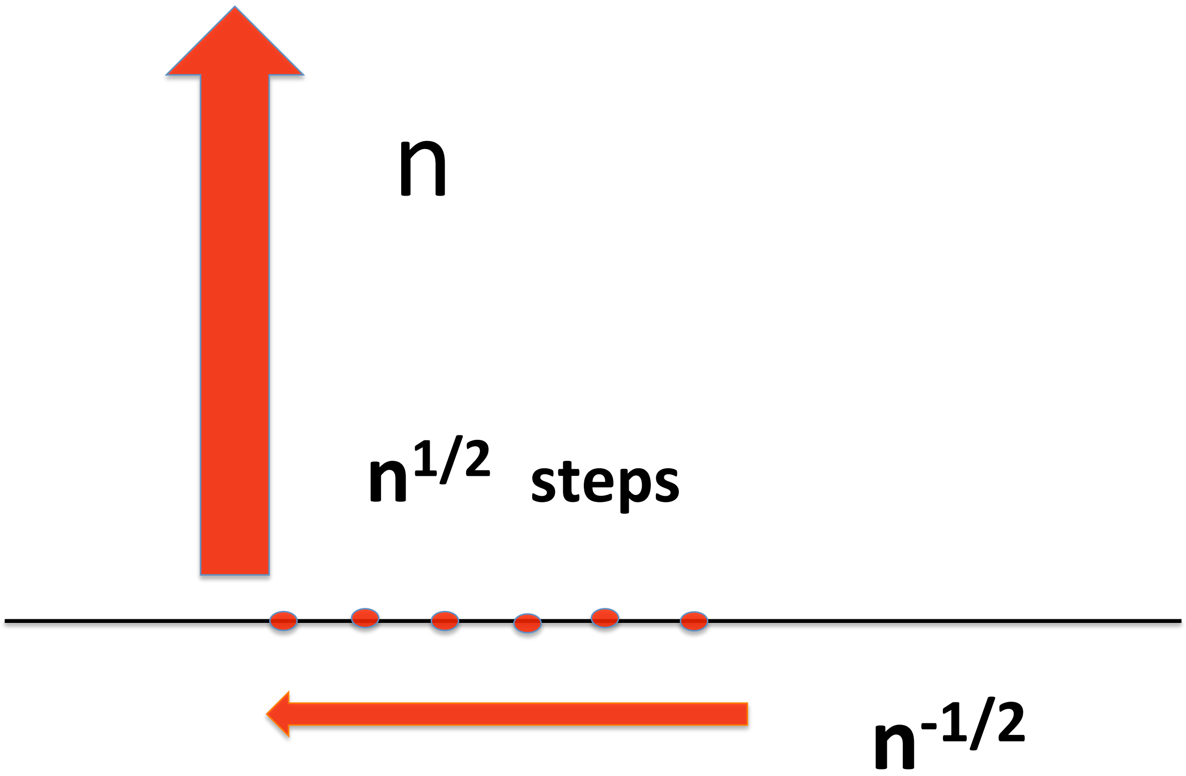
for a random covariate process V_t.

- More precisely, we will work on
  - a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathrm{F}, P)$,
  - $\mathrm{F} = (\mathcal{F}_t)_{t \in \hat{I}}$ being a filtration on $(\Omega, \mathcal{F})$, where $\hat{I} = [\hat{T}_0, T_1] \supset I$ and $n \in \mathbb{N}$.
  - For each $n \in \mathbb{N}$ and $\theta \in \Theta$, $(g^n(t, \theta))_{t \in I}$ is a d-dimensional predictable process,
  - $(K^n(t, s, \theta))_{s \in [\hat{T}_0, t)}$ is a $\mathsf{d} \times \mathsf{d}_0$ matrix-valued optional process for $t \in I$, $\mathcal{I}_0 = \{1, ..., \mathsf{d}_0\}$, and
  - $(X^n_t)_{t \in \hat{I}}$ is a d_0-dimensional F-adapted right-continuous increasing process on $\mathcal{B}$.
- The multivariate point process N^n is compensated by the process $(\int_{T_0}^t n\lambda^n(s, \theta) ds)_{t \in I}$ when θ is the true value of the unknown parameter.
- No common jumps of different elements of N^n

Our model

- **regression of the intensities to covariate processes and their history**
- **finite time horizon and the intensities of point processes tends to ∞ —- non-ergodic statistics**

Locally Poissonian and Globally non-ergodic model



Local model such as $\lambda_\alpha^{LA}(S_t)$ becomes a fibre.

Quasi Likelihood Analysis (QLA)

- **Ibragimov-Hasminskii and Kutoyants' program**
  + polynomial type large deviation inequality
  = Quasi likelihood analysis:
  - **ergodic / non-ergodic**
  - **limit theorems for QMLE and QBE**
  - **convergence of moments**
  - **Y. AISM 2011**

Recall Point Process Regression Model

$\boldsymbol{N^n = (N^{n,\alpha})_{\alpha \in \mathcal{I}}}$ **has an intensity process** $\boldsymbol{n\lambda^n(t,\theta)}$ **defined by**

$$\boldsymbol{\lambda^n(t,\theta) = g^n(t,\theta) + \int_{\hat{T}_0}^{t-} K^n(t,s,\theta)dX_s^n,}$$

Quasi likelihood

- **We shall consider estimation for the unknown parameter θ.**
- **Observations**

$$(N_t^{n,\alpha})_{t\in I,\alpha\in\mathcal{I}}, \quad (X_t^{n,\beta})_{t\in\hat{I},\beta\in\mathcal{I}_0},$$
$$(g^{n,\alpha}(t,\theta))_{t\in I,\alpha\in\mathcal{I},\theta\in\Theta},$$
$$(K_\beta^{n,\alpha}(t,s,\theta))_{t\in I,s\in[\hat{T}_0,t),\alpha\in\mathcal{I},\beta\in\mathcal{I}_0,\theta\in\Theta}.$$

This is the case, for example, when $g^{n,\alpha}(t,\theta)$ is a function of θ and some observable covariate process:

$$g^{n,\alpha}(t,\theta) = g^{\alpha}(t,V_t,\theta) \quad \text{with observable } V_t$$

- **The quasi log likelihood is given by**

$$\boldsymbol{l_n(\theta) = \sum_{\alpha \in \mathcal{I}} \left(\int_{T_0}^{T_1} \log[n\lambda^{n,\alpha}(t,\theta)] dN_t^{n,\alpha} - \int_{T_0}^{T_1} [n\lambda^{n,\alpha}(t,\theta) - 1] dt \right)}$$

for observed point process N^n. Obviously, "-1" in the second integral can be eliminated for maximization. The factor "n" in the first integral is also unnecessary. Thus we can use

$$\boldsymbol{\ell_n(\theta) = \sum_{\alpha \in \mathcal{I}} \left(\int_{T_0}^{T_1} \log \lambda^{n,\alpha}(t,\theta) dN_t^{n,\alpha} - \int_{T_0}^{T_1} n\lambda^{n,\alpha}(t,\theta) dt \right)} \tag{2}$$

instead of $l_n(\theta)$.

Statistical random field $\mathbb{Z}_n$

We shall work with the statistical random field

$$\mathbb{H}_n(\theta) = \ell_n(\theta)$$

on Θ and apply the frame of the quasi likelihood analysis.The random fields $\mathbb{Z}_n$ is defined on $\mathbb{U}_n = \{u \in \mathbb{R}^{\mathsf{p}};\ \theta_u \in \Theta\}$, $\theta_u = \theta^* + n^{-1/2}u$, by

$$\begin{aligned} \mathbb{Z}_n(u) &= \exp\big(\mathbb{H}_n(\theta_u) - \mathbb{H}_n(\theta^*)\big) \\ &= \exp\Bigg(\sum_{\alpha=1}^{\mathsf{d}} \int_{T_0}^{T_1} \log \frac{\lambda^{n,\alpha}(t,\theta_u)}{\lambda^{n,\alpha}(t,\theta^*)}\, dN_t^{n,\alpha} \\ &\qquad - \sum_{\alpha=1}^{\mathsf{d}} \int_{T_0}^{T_1} n\big[\lambda^{n,\alpha}(t,\theta_u) - \lambda^{n,\alpha}(t,\theta^*)\big]dt\Bigg). \end{aligned}$$

Assumptions

- **Assume that the boundary of Θ is good and that the function $\Theta \ni \theta \mapsto \lambda^n(t, \theta)$ has continuous extension to $\bar{\Theta}$ when the QMLE is discusses.**
- **Let ε be a positive number less than 1/2.**
- $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$

Assumptions

[B1]$_{\bar{j}}$ For each $n \in \bar{\mathbb{N}}$, $K^n(t,s,\theta)$ is an $\mathbb{R}_+^{\mathsf{d}} \otimes \mathbb{R}_+^{\mathsf{d}_0}$-valued $\mathcal{F} \times \mathbb{B}(J) \times \mathbb{B}(\Theta)$-measurable function satisfying the following conditions.

(i) For each $(n,t,\theta) \in \mathbb{N} \times I \times \Theta$, the process $[\hat{T}_0, t) \ni s \mapsto K^n(t,s,\theta)$ is $(\mathcal{F}_s)_{s\in[\hat{T}_0,t)}$-optional.

(ii) For each $(n,t,s) \in \bar{\mathbb{N}} \times J$, the mapping $\Theta \ni \theta \mapsto K^n(t,s,\theta)$ is $\bar{j}$ times differentialble a.s., $\sup_{(s,\theta)\in[\hat{T}_0,t)\times\Theta} |\partial_\theta^j K^n(t,s,\theta)| < \infty$ a.s. for $t \in I$, and

$$\sum_{j=0}^{\bar{j}} \sup_{(n,s,t)\in\bar{\mathbb{N}}\times J} \sup_{\theta\in\Theta} \|\partial_\theta^i K^n(t,s,\theta)\|_p < \infty$$

for every $p > 1$.

(iii) For each $(n,t,\theta) \in \mathbb{N} \times I \times \Theta$, the mappings $[\hat{T}_0, t) \ni s \mapsto \partial_\theta^i K^n(t,s,\theta)$ $(i = 0,1)$ are differen-tialble a.s., $\sup_{(s,\theta)\in[\hat{T}_0,t)\times\Theta} |\partial_s \partial_\theta^i K^n(t,s,\theta)| < \infty$ a.s. for $t \in I$, and

$$\sup_{(n,t,\theta)\in\mathbb{N}\times I\times\Theta} \sum_{i=0}^{1} \int_{\hat{T}_0}^{t} \|\partial_s \partial_\theta^i K^n(t,s,\theta)\|_p ds < \infty$$

for every $p > 1$.

(iv) For every $p > 1$,

$$n^\varepsilon \sum_{j=0}^{1} \sup_{(t,s)\in J,\theta\in\Theta} \left\| \partial_\theta^j K^n(t,s,\theta) - \partial_\theta^j K^\infty(t,s,\theta) \right\|_p \to 0$$

as $n \to \infty$.

[B2]$_{\bar{j}}$ For each $(\alpha, n) \in \mathcal{I} \times \bar{\mathbb{N}}$, $g^{n,\alpha}(t,\theta)$ is an nonnegative $\mathcal{F} \times \mathbb{B}(I) \times \mathbb{B}(\Theta)$-measurable function for which the following conditions are fulfilled.

(i) For each $(n, \alpha, \theta) \in \mathbb{N} \times \mathcal{I} \times \Theta$, the process $(g^{n,\alpha}(t,\theta))_{t \in I}$ is predictable.

(ii) For each $(n,t) \in \bar{\mathbb{N}} \times I$, the mapping $\Theta \ni \theta \mapsto g^n(t,\theta)$ is $\bar{j}$ times differentiable a.s. and

$$\sum_{j=0}^{\bar{j}} \sup_{(n,t) \in \bar{\mathbb{N}} \times I} \sup_{\theta \in \Theta} \left\| (\partial_\theta)^j g^n(t,\theta) \right\|_p < \infty$$

for every $p > 1$.

(iii) For every $p > 1$,

$$n^\varepsilon \sum_{j=0}^{1} \sup_{t \in I} \sup_{\theta \in \Theta} \left\| \partial_\theta^j g^n(t,\theta) - \partial_\theta^j g^\infty(t,\theta) \right\|_p \to 0$$

as $n \to \infty$.

[B3] For each $n \in \mathbb{N}$ and $\beta \in \mathcal{I}_0$, $(X_t^{n,\beta})_{t\in\hat{I}}$ is a non-decreasing $(\mathcal{F}_t)_{t\in\hat{I}}$-adapted process, and for each $\beta \in \mathcal{I}_0$, there exists a non-decreasing process $(X_t^{\infty,\beta})_{t\in I}$ such that

$$\sup_{(n,t)\in\mathbb{N}\times\hat{I}} \left\|X_t^{n,\beta}\right\|_p < \infty \text{ and}$$
$$n^{\varepsilon} \sup_{t\in\hat{I}} \left\|X_t^{n,\beta} - X_t^{\infty,\beta}\right\|_p \to 0$$

as $n \to \infty$, for every $p > 1$. ($\hat{I} = [\hat{T}_0, T_1]$.)

[B4] For each $(\omega, n, \alpha, t, \theta) \in \Omega \times \mathbb{N} \times \mathcal{I} \times I \times \Theta$, $\lambda^{n,\alpha}(t,\theta) = 0$ if and only if $\lambda^{n,\alpha}(t,\theta) = 0$, and

$$\sup_{(n,t,\theta)\in I\times\Theta} \left\|\lambda^{n,\alpha}(t,\theta)^{-1} 1_{\{\lambda^{n,\alpha}(t,\theta)\neq 0\}}\right\|_p < \infty$$

for every $p > 1$ and $\alpha \in \mathcal{I}$.

Information matrix and Limit intensity process

- Let

$$\Gamma = \sum_{\alpha \in \mathcal{I}} \int_{T_0}^{T_1} (\partial_\theta \lambda^{\infty,\alpha})^{\otimes 2} (\lambda^{\infty,\alpha})^{-1} (t, \theta^*) dt,$$

  where

$$\lambda^{\infty,\alpha}(t,\theta) = g^{\infty,\alpha}(t,\theta) + \int_{\hat{T}_0}^{t-} K_\beta^{\infty,\alpha}(t,s,\theta) dX_s^{\infty,\beta} \tag{3}$$

  for $t \in I$.

- $\lambda^{\infty,\alpha}(t,\theta)$ is possibly random.

Key index

$$\mathbb{Y}(\theta) := -\sum_{\alpha=1}^{\mathsf{d}} \int_{T_0}^{T_1} \Big[\lambda^{\infty,\alpha}(t,\theta) - \lambda^{\infty,\alpha}(t,\theta^*) - \log \frac{\lambda^{\infty,\alpha}(t,\theta)}{\lambda^{\infty,\alpha}(t,\theta^*)} \lambda^{\infty,\alpha}(t,\theta^*)\Big] dt \tag{4}$$

$$\chi_0 := \inf_{\theta\in\Theta\setminus\{\theta^*\}} \frac{-\mathbb{Y}(\theta)}{|\theta-\theta^*|^2}.$$

The nondegeneracy of the key index χ_0 will play an essential role in our argument.

[B5] For every $L > 0$, there exists a constant C_L such that

$$P[\chi_0 < r^{-1}] \leq \frac{C_L}{r^L} \quad (\forall r > 0).$$

Polynomial type large deviation inequality

Theorem 1. (Polynomial type large deviation inequality) Suppose that Conditions $[B1]_4$, $[B2]_4$, $[B3]$, $[B4]$ and $[B5]$ are fulfilled. Then, for every $L > 0$, there exists a constant C_L such that

$$P\left[\sup_{u\in\mathbb{V}_n(r)} \mathbb{Z}_n(u) \geq e^{-r}\right] \leq \frac{C_L}{r^L}$$

for all $r > 0$ and all $n \in \mathbb{N}$, where $\mathbb{V}_n(r) = \{u \in \mathbb{U}_n;\ |u| \geq r\}$.

Quasi likelihood analysis

Denote by $C_{\uparrow}(\mathbb{R}^{\mathsf{p}})$ the set of continuous functions $f : \mathbb{R}^{\mathsf{p}} \longrightarrow \mathbb{R}$ at most polynomial growth.

Theorem 2. Suppose that Conditions $[B1]_4$, $[B2]_4$, $[B3]$, $[B4]$ and $[B5]$ are fulfilled. Then

(a) $\sqrt{n}(\hat{\theta}_n - \theta^*) \longrightarrow^{d_s} \Gamma^{-1/2}\zeta$ **as** $n \to \infty$.

(b) $E\big[f\big(\sqrt{n}(\hat{\theta}_n - \theta^*)\big)\big] \longrightarrow \mathbb{E}\big[f\big(\Gamma^{-1/2}\zeta\big)\big]$ **as** $n \to \infty$ **for all** $f \in C_{\uparrow}(\mathbb{R}^{\mathsf{p}})$.

Theorem 3. Suppose that Conditions $[B1]_4$, $[B2]_4$, $[B3]$, $[B4]$ and $[B5]$ are fulfilled. Then

(a) $\sqrt{n}(\tilde{\theta}_n - \theta^*) \longrightarrow^{d_s} \Gamma^{-1/2}\zeta$ **as** $n \to \infty$.

(b) $E\big[f\big(\sqrt{n}(\tilde{\theta}_n - \theta^*)\big)\big] \longrightarrow \mathbb{E}\big[f\big(\Gamma^{-1/2}\zeta\big)\big]$ **as** $n \to \infty$ **for all** $f \in C_{\uparrow}(\mathbb{R}^{\mathsf{p}})$.

Example: A point process driven by a diffusion process

$$\lambda^n(t,\theta) = \lambda^\infty(t,\theta) = g(V_t,\theta)$$

for $t \in I$.

- V_t: a non-degenerate multi-dimensional diffusion process as the covariate
- For the non-degeneracy of χ_0, there are
  - an analytic criterion
  - a geometric criterion.
  - cf. Uchida-Y (SPA 2013)

Support function

Let

$$Q(x,\theta,\theta^*) = g(x,\theta)^{-1}g(x,\theta^*) - 1 - \log\left(g(x,\theta)^{-1}g(x,\theta^*)\right)$$

then

$$-2\mathbb{Y}(\theta) = \frac{1}{T}\int_0^T Q(V_t,\theta,\theta^*)g(V_t,\theta^*)dt.$$

A support function f is a function such that

$$Q(x,\theta,\theta^*)g(x,\theta^*)|\theta-\theta^*|^{-2} \geq |f(x,\theta)|^{\varrho},$$

for a constant $\varrho \in (0,\infty)$. Recall

$$\chi_0 = \inf_{\theta\neq\theta^*}\frac{-\mathbb{Y}(\theta)}{|\theta-\theta^*|^2} \geq \inf_{\theta\neq\theta^*}\frac{1}{2T}\int_0^T |f(V_t,\theta)|^{\varrho}dt.$$

Analytic criterion: nondegeneracy of a tensor field

- For simplicity, let $\mathsf{d} = 1$ and suppose that V is a nondegenerate Itô process.
- Suppose that $\mathcal{X}_0$ is a neighborhood of compact supp$\mathcal{L}\{V_0\}$, and that Θ is compact.
- For each $(x_0, \theta) \in \mathcal{X}_0 \times \Theta$, $\max_{j=0,...,J-1} \left|\partial_x^j f(x_0, \theta)\right| > 0$.

Then [B5] holds.

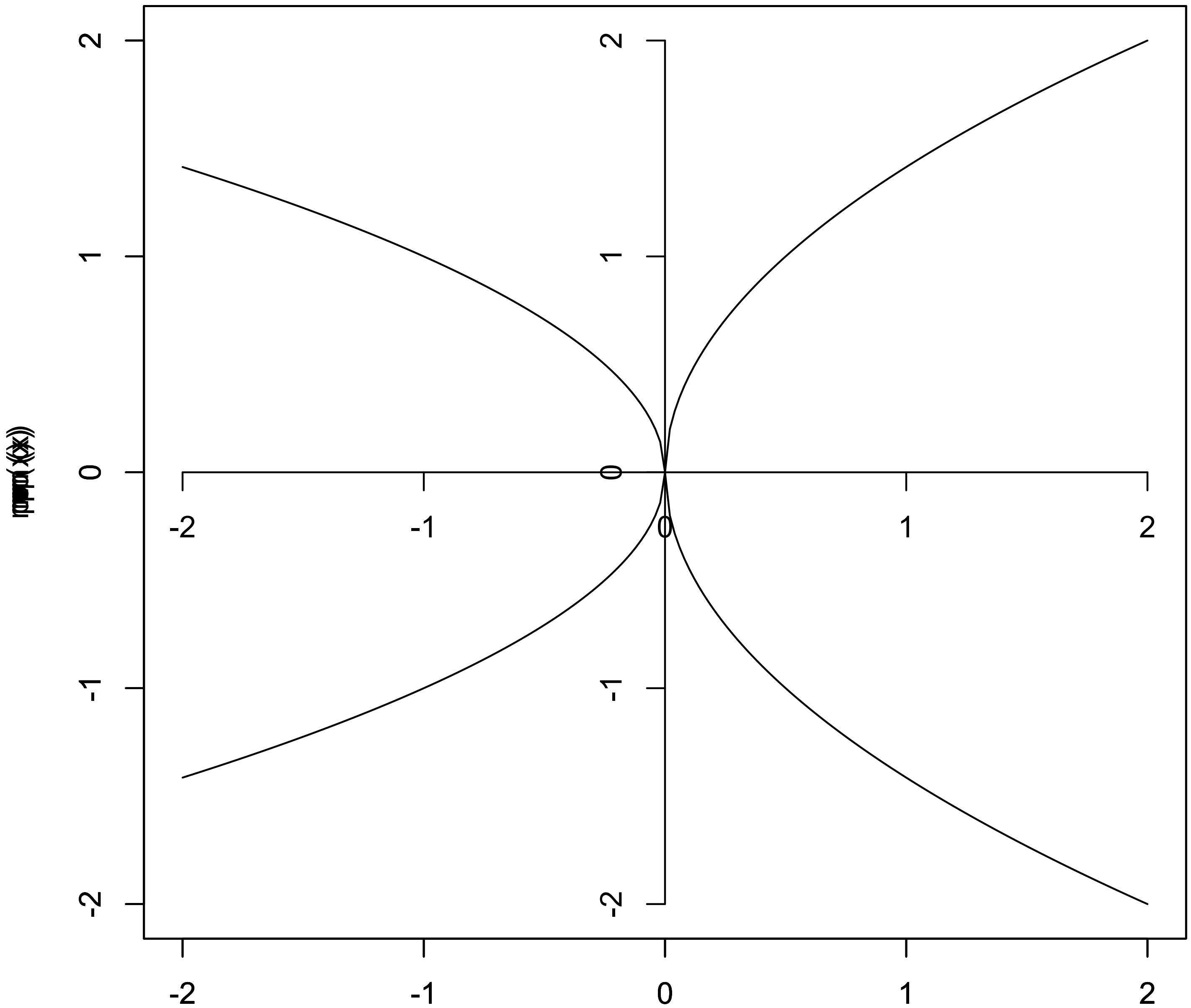
- Remarks.
  - Similar condition in the multi-dimensional case.
  - It is possible to give a condition for a degenerate diffusion on manifold. However the condition becomes much more complicated. (Uchida and Y LeMans2009, ISM RM2011, Paris2012)

Geometric criterion

- **Example.**
  - $f(x,\theta) = x_1 x_2 (x_1 - \theta_1 x_2^2)(\theta_2 x_1 + x_2^2)$
  - $V = (V_t) = (V_{1,t}, V_{2,t})$: **a nondegenerate diffusion with uniform initial distribution on** $\mathrm{supp}\mathcal{L}\{V_0\} = \{0\} \times [0,1]$.
  - **Show**

$$P\left[\inf_{\theta} \int_0^1 |f(V_t,\theta)|^2 dt < \frac{1}{r}\right] \leq \frac{C_L}{r^L}.$$

  - **The null set** $\{f = 0\}$ **is not a regular submanifold.**


-2
-1
0
1
2

Geometric criterion

[A3′] supp$\mathcal{L}\{V_0\}$ is compact, there exists a function $f : U \times \Theta \longrightarrow \mathbb{R}$ for some open neighborhood U of supp$\mathcal{L}\{X_0\}$ and the following conditions are satisfied.

(i) For some $\varrho \in (0, \infty)$, $Q(x, \theta, \theta^*)|\theta - \theta^*|^{-2} \geq |f(x, \theta)|^{\varrho}$ for all $(x, \theta) \in U \times (\Theta \setminus \{\theta^*\})$.

(ii) For each $x_0 \in U$, there exist a neighborhood B in U of x_0 and a covering $\{\Theta_k\}_{k=1,...,\bar{k}}$ of Θ such that for each $k = 1, ..., \bar{k}$, there exist $\xi_0 \in \mathbb{S}$, $J \in \mathbb{N}$, some positive numbers M, c, ϵ_0, K_j $(j = 1, ..., J)$ and some functions $\Psi_j : P^{\perp}_{\xi_0} B \times \Theta_k \longrightarrow \mathbb{R}$ such that

(a) each function $P^{\perp}_{\xi_0} B \ni y \mapsto \Psi_j(y, \theta) \in \mathbb{R}$ is M-Lipschitz continuous for all $\theta \in \Theta_k$,

(b) for $(x,\theta) \in B \times \Theta_k$,

$$|f(x,\theta)| \geq c \prod_{j=1}^{J} \left(|\xi_0 \cdot x - \Psi_j(P_{\xi_0}^{\perp} x, \theta)| \wedge \epsilon_0\right)^{K_j}.$$

Remarks

- In $[A3']$, $\bar{k}$ may depend on x_0.
- The null set

$$\{x \in B;\ f(x,\theta) = 0\} \subset \bigcup_{j=1}^{J} \{x \in B;\ \xi_0 \cdot x = \Psi_j(P_{\xi_0}^{\perp} x, \theta)\}$$

under $[A3']$(ii), that is, the graph of the functions Ψ_j covers locally the null set of f.

Theorem 4. (Uchida-Y arXiv2012 , SPA2013)

$[A3']$ + nondegenerate Itô process $V \Rightarrow [B5]$,

and hence QLA.

Example: A lookback regression

$$\lambda^n(t,\theta) = \lambda^\infty(t,\theta) \; = \; g(t,\gamma) + \int_0^t a e^{-b(t-s)} V_s ds$$

for $t \in I$.

- V_s: a positive non-degenerate diffusion process
- $g(t,\gamma)$: a polynomial taking non-negative values on the interval I
- (γ, a, b): unknown parameters
- The non-degeneracy of χ_0 is not trivial but provable. The exponential kernel is not essential.
- A multi-dimensional extension is possible.

Example: A non-stationary Hawkes process

- **The parametric model of two-dimensional Hawekes process with intensity process**

$$\lambda^n(t,\theta) = g(t,\gamma) + \int_{\hat{T}_0}^{t-} e^{-b(t-s)} A n^{-1} dN_s^n \quad (5)$$

with $\theta = (\gamma, b, A)$.

- **$g_t = g(t,\gamma)$ is an $\mathbb{R}^2$-valued polynomial in t.(inter-day trend)**
- **The non-degeneracy of χ_0 can be proved. Ogihara-Y (arXiv 2015)**

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