On the real zeros of random trigonometric polynomials

Jürgen Angst

Université de Rennes 1

Workshop - Numerical methods for algebraic curves
Rennes, February 19, 2018

Joint work with

- Guillaume Poly, Université de Rennes 1.
- Viet-Hung Pham, Vietnamese Institute of Advances Studies, Hanoï.
- Federico Dalmao, Universidad de la República de Uruguay, Salto.

We consider a polynomial $P_d \in \mathbb{R}[X]$,

$$P_d(X) = a_0 + a_1 X + \ldots + a_d X^d,$$

whose coefficients a_k are i.i.d. random variables that are centered with unit variance i.e.

$$\mathbb{E}[a_k] = 0, \quad \mathbb{E}[|a_k|^2] = 1.$$

Some natural questions

Can we estimate $N_d = \#Z_d$ where $Z_d := \{t \in \mathbb{R},\ P(t) = 0\}$? For a fixed degree d ? Asymptotically ? In mean ? almost surely ? Are the real zeros localized ?

Theorem (Kac, 1943)

If $a_k \sim \mathcal{N}(0,1)$ for all k's, then as d tends to infinity

$$\mathbb{E}[N_d] = \frac{2}{\pi}\log(d) + o\left(\log(d)\right).$$

Theorem (Erdös–Offord, 1956)

If $a_k \sim B(\pm 1, 1/2)$ for all k's, then as d tends to infinity

$$N_d = \frac{2}{\pi}\log(d) + o\left(\log(d)^{2/3}\log\log(d)\right)$$

with probability $1 - o\left(\frac{1}{\sqrt{\log\log(d)}}\right)$.

Theorem (Ibraghimov–Maslova, 1971)

If the a_k are centered with unit variance and in the domain of attraction of the normal law, then as d tends to infinity

$$\mathbb{E}[N_d] = \frac{2}{\pi}\log(d) + o\left(\log(d)\right).$$

Such an asymptotics is called **universal** in the sense that it does not depend on the particular law of a_k.

Theorem (Nguyen–Nguyen–Vu, 2015)

If the a_k are centered with unit variance and admit a finite moment of order $(2+\varepsilon)$, then as d goes to infinity

$$\mathbb{E}[N_d] = \frac{2}{\pi}\log(d) + O(1).$$

Tao–Vu (2013) : universality of the correlation functions.

Do–Nguyen–Vu (2016), Kabluchko–Flasche (2018) : universality of the mean number of real zeros of random polynomials of the form $P_d(X) = \sum_{k=0}^{d} a_k c_k X^k$.

The real zeros are localized near ± 1, in fact for all fixed $\eta > 0$

$$\mathbb{E}[\#\{t \in Z_d \cap [\pm 1 - \eta, \pm 1 + \eta]^c] = O(1).$$

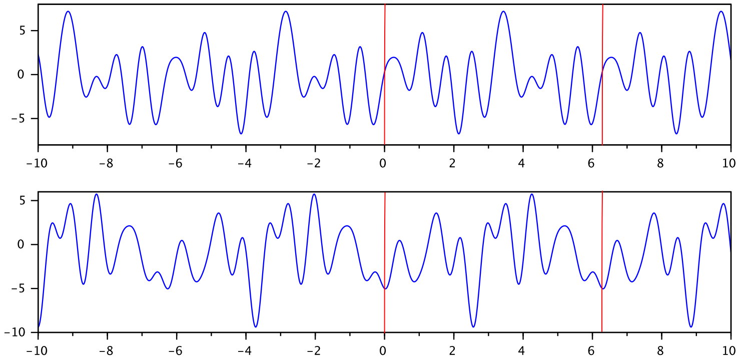
We will consider here a sum of the type

$$f_n(t) = \sum_{\substack{k\in\mathbb{Z}^d \\ |k|\leq n}} a_k\, e^{2\pi i k\cdot t}, \quad t \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d,$$

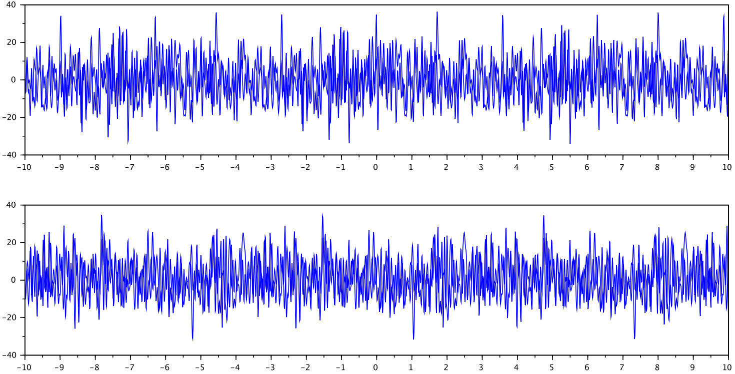
with the same symmetry constraint $\overline{a_k} = a_{-k}$, so that f_n is real-valued.

The random coefficients a_k are supposed to be independent, identically distributed, centered with unit variance

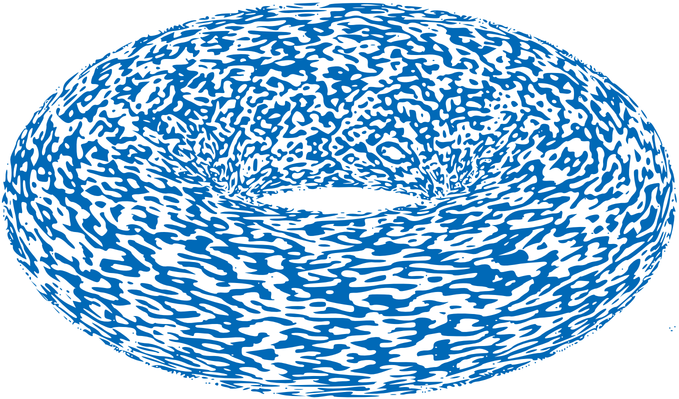
$$\mathbb{E}[a_k] = 0, \qquad \mathbb{E}[|a_k|^2] = 1.$$



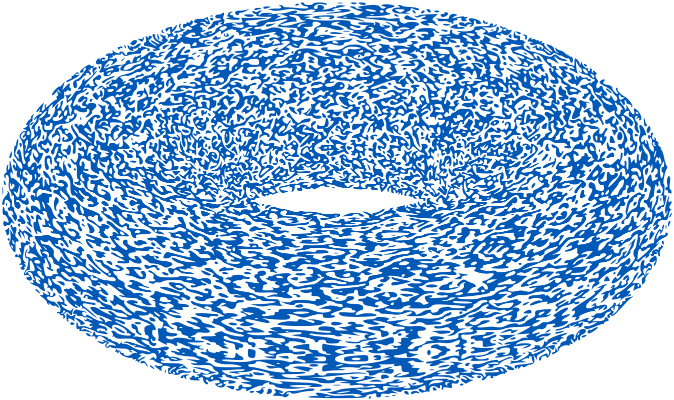
two realizations of f_n in dimension $d = 1$ and degree $n = 10$ for Gaussian and Bernoulli coefficients.



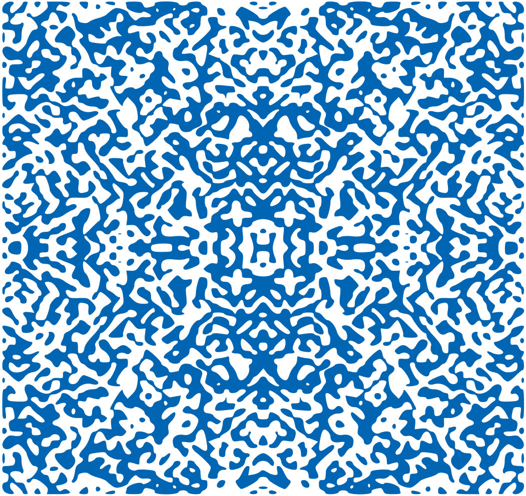
two realizations of f_n in dimension $d = 1$ and degree $n = 100$ for Gaussian and Bernoulli coefficients.



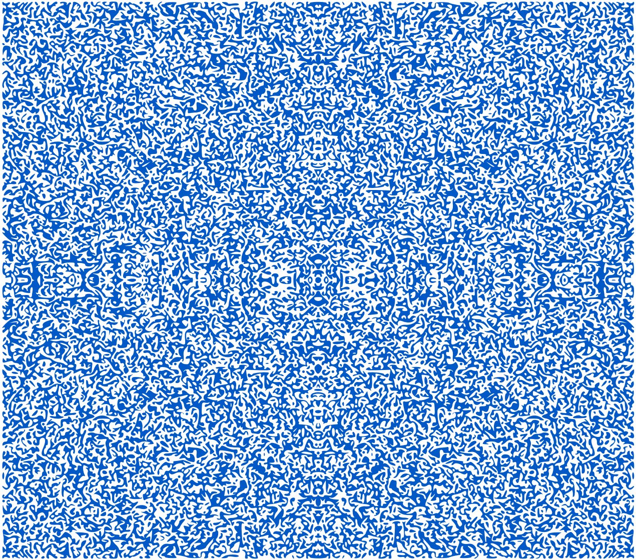
a realization of the interface between $\{f_n > 0\}$ (white) and $\{f_n < 0\}$ (blue) for $d = 2$, $n = 30$, Gaussian coefficients



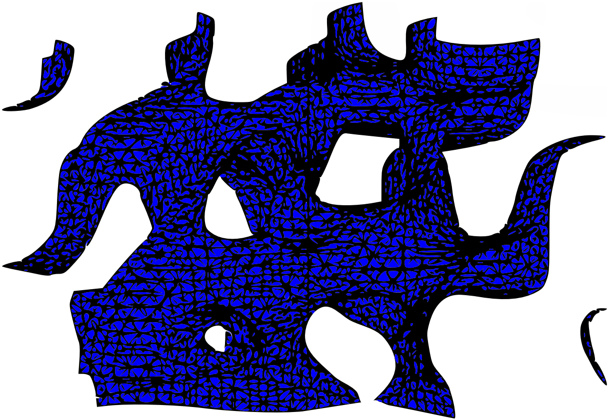
a realization of the interface between $\{f_n > 0\}$ (white) and $\{f_n < 0\}$ (blue) for $d = 2$, $n = 100$, Gaussian coefficients



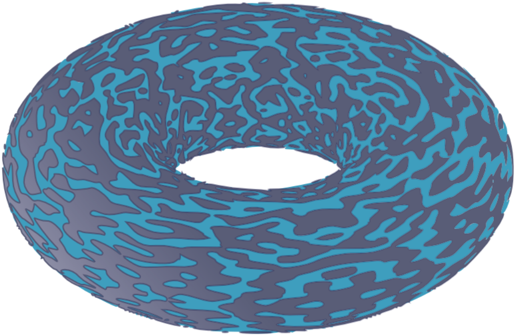
on the unfolded torus in dimension $d = 2$, degree $n = 40$, Bernoulli coefficients

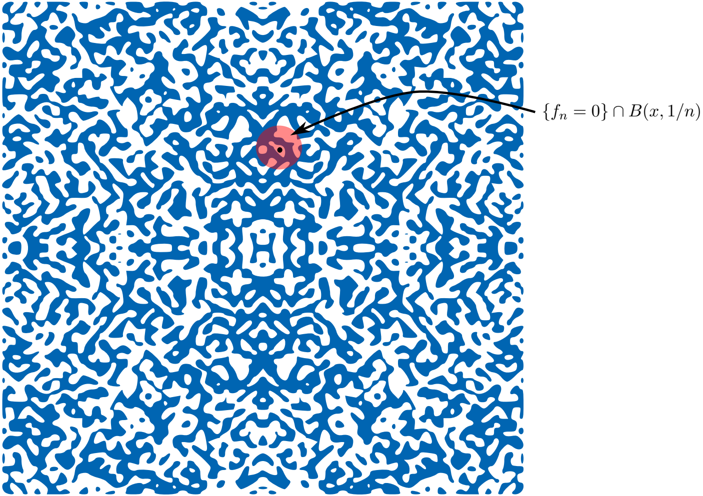


$d = 2$, degree $n = 200$, Bernoulli coefficients



$d = 3$, degree $n = 20$, Bernoulli coefficients





Local study of $\mathcal{H}^{d-1}\left(\{f_n = 0\} \cap B\left(x, \frac{1}{n}\right)\right)$

Theorem (A.–Pham–Poly, 2016)

If the coefficients a_k are independent, identically distributed, centered with unit variance, then for all $x \in \mathbb{T}^d$, as n goes to infinity, the sequence of random variables

$$n^{d-1} \times \mathcal{H}^{d-1}\left(\{f_n = 0\} \cap B\left(x, \frac{1}{n}\right)\right)$$

converges in distribution to an explicit random variable whose law does no depend on x, nor on the law of a_k.

Heuristics of the proof :

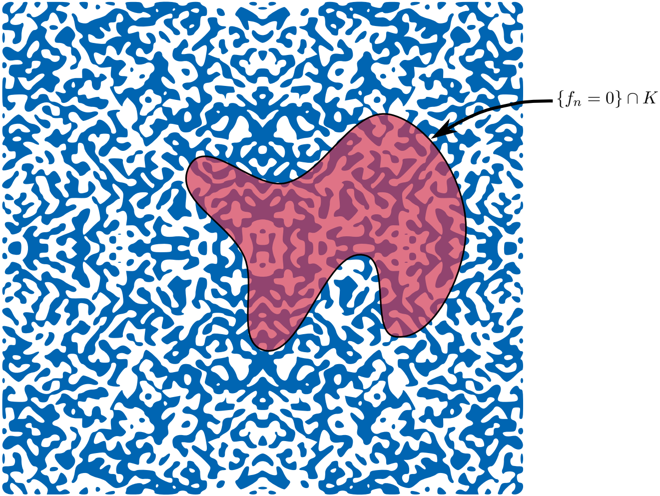
- The zeros of $f_n(t)$ identify with the ones of $(X_n(t))_{|t|\leq 1}$

$$X_n(t) := \frac{1}{n^{d/2}} \times f_n\left(x + \frac{t}{n}\right)$$

$$= \frac{1}{n^{d/2}} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|\leq n}} a_k\, e^{2\pi i k\cdot\left(x+\frac{t}{n}\right)}.$$

- The processes $(X_n(t))$ converge in distribution with respect to the C^1-topology to an universal Gaussian process.

- The nodal volume is a continuous functional with respect to the C^1-topology.



Global study of $\mathcal{H}^{d-1}\left(\{f_n = 0\} \cap K\right)$

Theorem (A.–Pham–Poly, 2016)

If the random coefficients a_k are independent, identically distributed, centered with unit variance and if $K \subset \mathbb{T}^d$ is a compact set with a smooth boundary, then there exists a universal explicit constant C_d such that, as n goes to infinity

$$\lim_{n \to +\infty} \frac{1}{n} \times \mathbb{E}\left[\mathcal{H}^{d-1}\left(\{f_n = 0\} \cap K\right)\right] = C_d \times \mathcal{H}^d(K).$$

Heuristics of the proof :

- Divide the set K into microscopic boxes $(K_n^j)_j$ of size $1/n$.
- In each box K_n^j, we have local universality.
- We sum the local contributions...
- ... and exhibit a uniform upper bound

$$\sup_{n,j} \mathbb{E}\left[|\mathcal{H}^{d-1}\left(\{f_n = 0\} \cap K_j^n\right)|^{1+\alpha} \right] < +\infty.$$

We consider the trigonometric polynomial

$$f_n(t) := \sum_{1 \le k \le n} a_k \cos(kt) + b_k \sin(kt), \quad t \in \mathbb{R},$$

where the $(a_k)_{k \ge 1}$ and $(b_k)_{k \ge 1}$ are centered Gaussian variable with correlation $\rho : \mathbb{N} \to \mathbb{R}$, i.e.

$$\begin{aligned} \mathbb{E}[a_k a_\ell] &= \mathbb{E}[b_k b_\ell] =: \rho(|k-\ell|), \\ \mathbb{E}[a_k b_\ell] &= 0, \quad \forall k, \ell \in \mathbb{N}^*. \end{aligned}$$

Theorem (A.–Dalmao–Poly, 2017)

If the correlation function ρ satisfies some mild hypotheses

$$\lim_{n\to+\infty} \frac{1}{n} \times \mathbb{E}\left[\mathcal{H}^0\left(\{f_n = 0\} \cap K\right)\right] = \frac{1}{\pi\sqrt{3}} \times \mathcal{H}^1(K).$$

This is the exact same asymptotics as in the i.i.d. case

Theorem (A.–Poly 2015, Flasche, 2016)

If the coefficients a_k are i.i.d. centered with unit variance

$$\lim_{n\to+\infty} \frac{1}{n} \times \mathbb{E}\left[\mathcal{H}^0\left(\{f_n = 0\} \cap K\right)\right] = \frac{1}{\pi\sqrt{3}} \times \mathcal{H}^1(K).$$

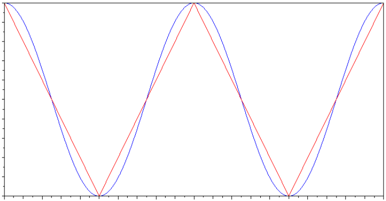
The asymptotics of the mean number of zeros is universal, but the asymptotics of the variance is not !

Theorem (Bally–Caramellino–Poly, 2017)

If the coefficients a_k are i.i.d. centered with unit variance, and have a density with respect to the Lebesgue measure

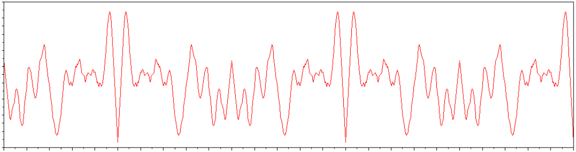
$$\lim_{n\to+\infty} \frac{1}{n} \times \mathrm{var}\left[\mathcal{H}^0\left(\{f_n = 0\} \cap [0,\pi]\right)\right] = C_{\mathcal{N}(0,1)} + \frac{1}{30}\left(\mathbb{E}[a_k^4] - 3\right).$$

Let ϕ be a C^0, $1-$periodic function, piecewise $C^{1+\alpha}$ Hölder.



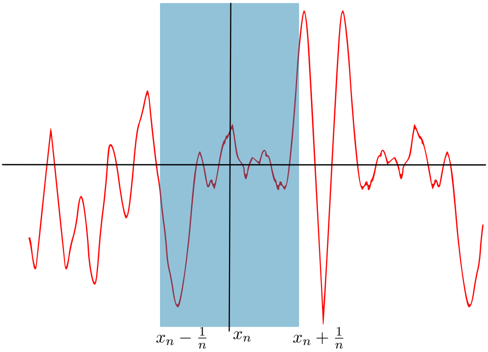
We consider the random periodic function

$$f_n(t) = \sum_{k=1}^{n} a_k \phi(kt).$$



As before, we look at the zeros of f_n in a small ball $B(x_n, 1/n)$, i.e we consider

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \, \phi\left(k\left(x_n + \frac{t}{n}\right)\right).$$



Theorem (A.–Poly, 2017)

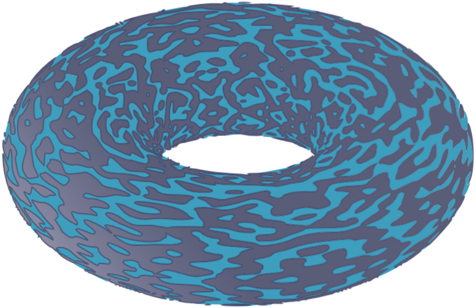
If the coefficients a_k are independent, identically distributed, centered with unit variance, with a finite fourth moment, and if $\alpha := \lim_{n\to+\infty} x_n \in \mathbb{R}\backslash\mathbb{Q}$ is diophantine, then the sequence

$$\mathcal{H}^0\left(\{f_n = 0\} \cap B\left(x_n, \frac{1}{n}\right)\right)$$

converge in distribution to a random variable whose law does not depend on α, nor the particular law of a_k.

Remark : if $\alpha \in \mathbb{Q}$, convergence holds but the limit is no more universal.

ANR research project on universality for random nodal domains



UNI RA NDOM N DOM AINS
VERSALITY ODAL

http://unirandom.univ-rennes1.fr