Curves and their Jacobians in the Monopole problem

Victor Enolski

National University of ‘Kyiv-Mohyla Academy‘

in collaboration with H.Braden (Edinburgh)

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Numerical methods for algebraic curves
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Movie: Level of energy density of charge 2 monopole in $\mathbb{R}^3$

Fix elliptic curve

$$\mathcal{C} = (\zeta, \eta): \qquad \eta^2 + \frac{K^2}{4}(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1) = 0$$

Find four solutions

$$\zeta_j(\mathbf{x}), \qquad j = 1, \dots 4$$

of quartic equation, **Atiyah-Ward constraint**

$$\eta = (x_2 + \imath x_1)\zeta^2 + 2x_3\zeta + x_2 - \imath x_1$$

Build four transcendents

$$\mu_j(\mathbf{x}) = \exp\left\{\int_{k'+\imath k}^{\zeta_j(\mathbf{x})} \frac{\mathrm{d}\zeta}{\eta}\left(\zeta^2 - \frac{2E-K}{K}\right)\right\}, \quad j = 1, \dots, 4$$

A level of energy density $\mathcal{E}(\mathbf{x})$ of charge 2 monopole is built in terms quantities $\zeta_j(\mathbf{x})$,$\mu_j(\mathbf{x})$ by certain explicit formula explained below

Yang-Mils action

Gauge group $U(1) \times SU(2) \times SU(3) \subset SU(N)$
Yang-Mils free action

$$S = \frac{1}{2e^2} \int \mathrm{d}^4 x \, \mathrm{Tr} \, F_{i,j} \, F^{i,j}, \quad i, j = 1, \ldots, 4$$

Yang-Mils field strengths

$$F_{i,j} = \partial_{x_i} a_j - \partial_{x_j} a_i + [a_i, a_j]$$

Covariant derivative

$$D_i \Phi = \partial_i \Phi + [a_i, \Phi]$$

Finite action solutions

$$D_i F^{i,j} = 0$$

That's second order PDE

Instanton solution

Belavin, Polyakov, Schwartz, Tyupkin, 1975 (BPST) rewrote the action in $\mathbb{R}^4$ (imaginary time t) by completing the square

$$S_{\text{inst}} = \frac{1}{4e^2} \int \mathrm{d}^4 x \, \mathrm{Tr} \, (F_{i,j} \mp {}^*F^{i,j})^2 \pm \mathrm{Tr} \underbrace{F_{i,j} {}^*F^{i,j}}_{\text{Total derivative}}$$

where dual field strength are defined as ${}^*F_{i,j} = \frac{1}{2}\epsilon_{i,j,k,l}F^{k,l}$ and second term $\sim n \in \mathbb{N}$. Then

$$S_{\text{inst}} \geq \frac{8\pi^2}{e^2}|n|, \quad n \in \mathbb{N}$$

with equality iff

$$F_{i,j} = {}^*F_{i,j}, \quad n > 0; \quad \text{or} \quad F_{i,j} = -{}^*F_{i,j}, \quad n < 0$$

BPST found explicitly $n = 1$ solution in algebraic form

In what follows $SU(N) = SU(2)$

Static instanton called Non-Abelian Monopole

$$\frac{\partial}{\partial x_4} a_i = 0, \quad i = 1, \ldots, 4, \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$$

$$\text{Gauge fields :} \quad a_1(\mathbf{x}), a_2(\mathbf{x}), a_3(\mathbf{x}), \quad a_4(\mathbf{x}) = \Phi(\mathbf{x})$$

Density of the Yang-Mills-Higgs action

$$\sim \mathrm{Tr}\, F_{i,j} F^{i,j} + \mathrm{Tr}\, D_i \Phi D^i \Phi, \quad i, j = 1, 2, 3$$

Self-duality condition = **Bogomolny equations, 1976**

$$D_i \Phi = \pm \sum_{j,k} \epsilon_{ijk} F_{jk}, \quad i = 1, 2, 3$$

which should be solved at the boundary conditions

$$H = \sqrt{-\frac{1}{2} \mathrm{Tr}\, \Phi(\mathbf{x})^2} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Nahm(1980)-Hitchin(1983) theorem,

Exist two and only two orthonormalizable solutions, $\mathbf{v}_\mu(s,\mathbf{x})$ *to the*

$$\textbf{Weyl equation}: \quad \left(-\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}s} + \sum_{j=1}^{3}(T_j(s) + \imath x_j 1_n)\otimes\sigma_j\right)\mathbf{v}(s,\mathbf{x}) = 0$$

with $n \times n$ *matrices* $T_j(s)$ *satisfying to the*

$$\textbf{Nahm equation}: \quad \frac{\mathrm{d}\,T_i(s)}{\mathrm{d}s} = \frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j(s), T_k(s)]$$

$T_i(s)$ *are regular* $s \in (0,2)$ *have simple poles at* $s = 0, 2$*;*
$\mathrm{Res}_{s=0} T_i(s)$ *- n-dim. irreducible representation of* $SU(2)$*, also*

$$T_i(s) = -T_i^\dagger(s), \quad T_i(s) = T_i^\dagger(2-s)$$

Then monopole field $\Phi(\mathbf{x})_{\mu\nu}$ *is given as*

$$\Phi(\mathbf{x})_{\mu\nu} = \imath\int_0^2 s\mathbf{v}_\mu^\dagger(s,\mathbf{x})\,.\,\mathbf{v}_\nu(s,\mathbf{x})\mathrm{d}s,\ \mu,\nu = 1,2$$

And similar formula for gauges a_i

$$a_i(\mathbf{x})_{\mu\nu} = \imath \int_0^2 \mathbf{v}_\mu^\dagger(s, \mathbf{x}) \,.\, \partial_{x_i} \mathbf{v}_\nu(s, \mathbf{x}) \mathrm{d}s, \quad i = 1, 2, 3,$$

Hitchin solution to the Nahm equation (1982,1983)

Nahm equations admit Lax form:

$$\frac{\mathrm{d}A(s,\zeta)}{\mathrm{d}s} = [A(s,\zeta), M(s,\zeta)]$$
$$A(z,\zeta) = A_{-1}(s)\zeta^{-1} + A_0(s) + A_{+1}(s)\zeta,$$
$$M(s,\zeta) = \frac{1}{2}A_0(s) + \zeta A_{+1}(s)$$
$$A_{\pm 1}(s) = T_1(s) \pm \imath T_2(s), \quad A_0(s) = 2\imath T_3(s)$$

Condition

$$\det(A(s,\zeta) - \eta 1_n) = 0$$

yields the curve $\hat{\mathcal{C}} = (\zeta, \eta)$ of genus

$$g_{\widehat{\mathcal{C}}} = (n-1)^2$$

n-charge **monopole curve**

$$\eta^n + \alpha_1(\zeta)\eta^{n-1} + \ldots + \alpha_n(\zeta) = 0$$

$a_j(\zeta)$- polynomials in ζ of degree $2j$.

Further plan

- (I) Problems appearing at finding monopole curve
- (II) Calculation of monopole fields and energy density
- (III) The case of charge 2
- (IV) Further problems

Part I: Monopole curve

$$\eta^n + \alpha_1(\zeta)\eta^{n-1} + \ldots + \alpha_n(\zeta) = 0$$

$a_j(\zeta)$- polynomials in ζ of degree $2j$.

genus $g = (n-1)^2$

Hitchin constraints (1982,1983)

H1. $\mathcal{C}$ admits the involution

$$(\zeta,\eta) \to \left(-1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2\right)$$

H2. Let γ_∞ is the second kind normalized differential on $\mathcal{C}$

$$\gamma_\infty(P)_{P\to\infty_i} = \left(\frac{\rho_i}{\xi^2} + O(1)\right)\mathrm{d}\xi, \quad \oint_{\mathfrak{a}_k} \gamma_\infty = 0, \quad \rho = \mathrm{e}^{2\imath\pi/n}$$

Then its $\mathfrak{b}_k$-periods, $k = 1, ..., g$ are half-periods

$$\mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_n}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m},$$

$\mathbf{n}, \mathbf{m} \in \mathbb{Z}^g$- **Ercolani-Sinha vectors** [E.Ercolani, A.Sinha, 1989]
H3. Linear winding $\mathbf{U}s + \mathbf{K}$, $\mathbf{K}$- vector of Riemann constants, does not intersect theta-divisor inside the interval $(0,2)$, i.e.:

$$\theta(\mathbf{U}s + \mathbf{K}; \tau) \neq 0, \quad s \in (0,2)$$

Tetrahedral monopole

Hitchin,Manton,Murray, 1995 found charge 3 monopole curve of genus 4

$$\eta^3 + \zeta^6 + 5\sqrt{2}\zeta^3 - 1 = 0$$

The equation includes Kleinian polynomial $\zeta^6 + 5\sqrt{2}\zeta^3 - 1$ which is invariant under action of tetrahedral group.

The curve admits C_3 symmetry,

$$(\zeta, \eta) \longrightarrow (\rho\zeta, \rho\eta), \quad \rho = \mathrm{e}^{2\imath\pi/3}.$$

Extension on the above result

Let us try to extend this result to general curve with C_3 symmetry,

$$\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0, \quad \alpha, \beta, \gamma \in \mathbb{R}$$

Theorem [Braden & E, 2010] *For the family of trigonal curves*

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0$$

Hitchin constraints satisfy only and only for the following values of parameters χ *and* b

$$b = \pm 5\sqrt{2}, \qquad \chi = -\frac{1}{6}\frac{\Gamma(1/6)\Gamma(1/3)}{2^{1/6}\pi^{1/2}}$$

Below - the sketch of the prove

Wellstein (1899), Matsumoto (2000)

Trigonal curve $\widehat{\mathcal{C}}$: $w^3 = (z - \lambda_1) \dots (z - \lambda_6)$

Period matrix $\widehat{\tau} = \rho^2 \left(H + (\rho^2 - 1) \frac{\mathbf{X}\mathbf{X}^T}{\mathbf{X}^T H \mathbf{X}} \right), \quad \rho = \exp(2\imath\pi/3)$

$$H = \mathrm{diag}(1, 1, 1, -1),$$

$$\mathbf{X} = \left(\oint_{\mathfrak{a}_1} \frac{\mathrm{d}z}{w}, \dots, \oint_{\mathfrak{a}_4} \frac{\mathrm{d}z}{w} \right).$$

Homology basis by Wellstein (1899)

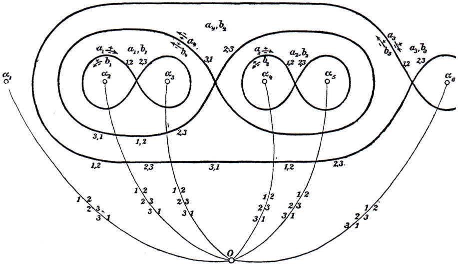


Fig. 1.

Solving H2 constraint

Proposition *For a pair of relatively prime integers* (m, n) *for which*

$$(m+n)(m-2n) < 0$$

a solution to **H2** *can be obtained as follows: solve for* t

$$\frac{2n-m}{m+n} = \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1,t\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1,1-t\right)}$$

Then

$$b = \frac{1-2t}{\sqrt{t(1-t)}}, \quad t = \frac{-b+\sqrt{b^2+4}}{2\sqrt{b^2+4}}$$

and

$$\chi^{1/3} = -(n+m)\frac{\alpha}{(1+\alpha^6)^{1/3}}{}_2F\left(\frac{1}{3},\frac{2}{3};1,t\right), \quad \alpha = t/(1-t)$$

Ramanujan hypergeometric relation

At $n = 1$ and $m = 0$ should be:

$$\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t\right)} = 2,$$

Amazingly

$$t = \frac{1}{2} - \frac{5\sqrt{3}}{18}, \quad b = 5\sqrt{2}$$

Ramanujan, 1915: Let r (signature) and $n \in \mathbb{N}$

$$\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)} = n\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-y\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; y\right)}.$$

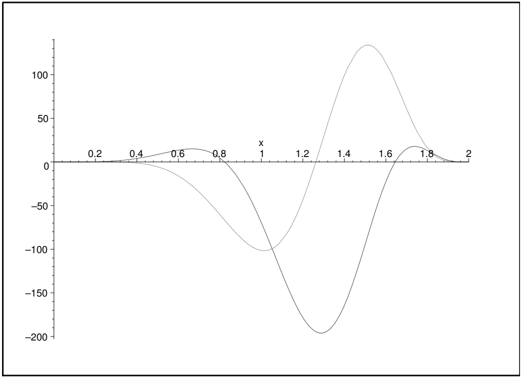
Then $\mathcal{P}(x, y) = 0$ is algebraic equation, find it!
Ramanujan theory for signature 3, $r = 3$, $n = 2$

$$(xy)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}(1-y)^{\frac{1}{3}} = 1$$

Set $y = \frac{1}{2}$ to obtain $b = 5\sqrt{2}$.
Other signatures: **Berndt** & **Bhargava** & **Garvan, 1995**

Tetrahedral monopole exists

Value $b = 5\sqrt{2}$ corresponds to $n = 1, m = 0$ - Check **H3**



Plot of the real and imaginary parts of the function $\theta(\mathbf{U}s + \mathbf{K})$, $s \in [0, 2]$

The case $b = -5\sqrt{2}$ is given by $n = m = 1$

Unramified cover

Our genus 4 curve $\widehat{\mathcal{C}}$ admits automorphism: $\sigma : (\zeta, \eta) \to (\rho\zeta, \rho\eta)$ and covers 3-sheetedly genus 2 curve $\mathcal{C}$.

$$\pi : \widehat{\mathcal{C}} \to \mathcal{C}$$

$$\widehat{\mathcal{C}}(\zeta, \eta) : \quad \eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0,$$

$$\mathcal{C}(\mu, \nu) : \quad \nu^2 = (\mu^3 + b)^2 + 4$$

$$\nu = \zeta^3 + 1/\zeta^3, \qquad \mu = -\eta/\zeta$$

Riemann-Hurwitz formula,

$$2 - 2\widehat{g} = 2N(1 - g) - B$$

tells that the cover is unramified, $N = 3, \widehat{g} = 4, g = 2 \to B = 0$.

Fay-Accola theorem, Fay-63, p.67

Theorem *For unramified cover* $\pi : \widehat{\mathcal{C}}(\zeta, \eta) \longrightarrow \mathcal{C}(x, y)$ *exists a basis in homology group* $(\mathfrak{a}_0, \ldots, \mathfrak{a}_3; \mathfrak{b}_0, \ldots, \mathfrak{b}_3)$ *admitting automorphism* σ*,*

$$\sigma \circ \mathfrak{a}_k = \mathfrak{a}_{k+1,\mathrm{mod}3}, \; \sigma \circ \mathfrak{b}_k = \mathfrak{b}_{k+1,\mathrm{mod}3}, \; k = 1, 2, 3,$$
$$\sigma \circ \mathfrak{a}_0 \sim \mathfrak{a}_0, \; \sigma \circ \mathfrak{b}_0 \sim \mathfrak{b}_0$$

Then remarkable factorization occurs

$$\frac{\theta(3z_1, z_2, z_2, z_2; \widehat{\tau})}{\theta(z_1, z_2; \tau)\theta(z_1 + 1/3, z_2; \tau)\theta(z_1 - 1/3, z_2; \tau)} = c$$

Here c *independent of* z_1, z_2*, period matrices are*

$$\widehat{\tau} = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix} \qquad \tau = \begin{pmatrix} \frac{1}{3}a & b \\ b & c + 2d \end{pmatrix}.$$

Humbert variety

Humbert variety H_{h^2}: period matrix τ of genus two curve $\mathcal{C}$ satisfies

$$q_1 + q_2\tau_{11} + q_3\tau_{12} + q_4\tau_{22} + q_5(\tau_{12}^2 - \tau_{11}\tau_{22}) = 0;$$

$$q_i \in \mathbb{Z}, \quad q_3^2 - 4(q_1q_5 + q_2q_4) = h^2, \quad h \in \mathbb{N}.$$

Then exists a symplectic transformation $\mathfrak{S}$

$$\mathfrak{S} : \tau \to \begin{pmatrix} T_1 & \frac{1}{h} \\ \frac{1}{h} & T_2 \end{pmatrix}, \quad h \in \mathbb{N}.$$

Here h - degree of the cover $\mathcal{C}$ over elliptic curve $\mathcal{E}$

$$\pi : \mathcal{C} \to \mathcal{E}.$$

In the case considered we got $h = 2$

15 components of H_4, Prinsheim, 1875

$$y^2 = x(1-x)(1-\lambda^2 x)(1-\mu^2 x)(1-\kappa^2 x)$$

$$2\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2 = 0 \quad \Leftrightarrow \quad \kappa^2 = \lambda^2\mu^2$$

$$\tau_{11} + 2\tau_{12}\tau_{22} = 0 \quad \Leftrightarrow \quad \kappa^2 - \lambda^2 = \mu^2(1-\lambda^2)$$

$$\tau_{11} + 2\tau_{12} - (\tau_{11}\tau_{22} - \tau_{12}^2) = 0 \quad \Leftrightarrow \quad \kappa^2(1-\mu^2) = \lambda^2(\kappa^2 - \mu^2)$$

$$2\tau_{12} - \tau_{22} = 0 \quad \Leftrightarrow \quad \mu^2 = \kappa^2\lambda^2$$

$$2\tau_{12} - \tau_{22} + \tau_{11}\tau_{22} - \tau_{12}^2 = 0 \quad \Leftrightarrow \quad \lambda^2 = \kappa^2\mu^2$$

$$\tau_{11} - \tau_{22} = 0 \quad \Leftrightarrow \quad \kappa^2 - \lambda^2 = \lambda^2(\kappa^2 - \mu^2)$$

$$\tau_{11} - \tau_{22} + \tau_{11}\tau_{22} - \tau_{12}^2 = 0, \quad \Leftrightarrow \quad \kappa^2 - \mu^2 = \mu^2(\kappa^2 - \lambda^2)$$

$$2\tau_{12} = 1 \quad \Leftrightarrow \quad \lambda^2 - \mu^2 = -\kappa^2(1-\lambda^2)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

H_4 components in our simplifications

Genus two period matrix in Fay-Accola reduction,

$$\begin{pmatrix} \frac{1}{3}a & b \\ b & c+2d \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}$$

is proved to be H_4-component

$$1 - \tau_{11} + \tau_{11}\tau_{22} - \tau_{12}^2 = 0$$

and mapped to the component

$$\tau_{12} = \frac{1}{2}$$

by symplectic transformation $\mathfrak{S}$

Outline of theta-transformations

$$\widehat{\tau} = \rho^2 H - (\rho - \rho^2)\frac{(\mathbf{n} + \rho^2 H\mathbf{m})(\mathbf{n} + \rho^2 H\mathbf{m})^T}{(\mathbf{n} + \rho^2 H\mathbf{m})^T H(\mathbf{n} + \rho^2 H\mathbf{m})}.$$

Wellstein

$$\Downarrow$$

$$\begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}$$

$$\Downarrow$$

Fay-Accola

$$\begin{pmatrix} \frac{1}{3}a & b \\ b & c + 2d \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} T & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{12T} \end{pmatrix}$$

Bolza D_6

H3 condition is reduced to

Proposition [Braden & E, 2009]

$$\theta(\mathbf{U}s + \mathbf{K}; \tau) = 0 \quad \text{at} \quad s \in (0, 2)$$

iff one from the following **3** conditions satisfies

$$\frac{\vartheta_3}{\vartheta_2}\left(y\sqrt{-3} + \varepsilon\frac{T}{3}|T\right) + (-1)^{\varepsilon}\frac{\vartheta_2}{\vartheta_3}\left(y + \varepsilon\frac{1}{3}|\frac{T}{3}\right) = 0$$

$$\varepsilon = 0, \pm 1, \quad y = \frac{1}{3}s(n+m), \quad T = \frac{2\sqrt{-3}(n+m)}{2n-m}$$

The solution $y = y(T)$ provides the answer.
We reduced problem in $(n, m) \in \mathbb{Z}^2$ to one variable T

A new θ-constant relation ?

$$\frac{\vartheta_3}{\vartheta_2}\left(\frac{\tau}{3}|\tau\right)=\frac{\vartheta_2}{\vartheta_3}\left(\frac{1}{3}|\frac{\tau}{3}\right)$$

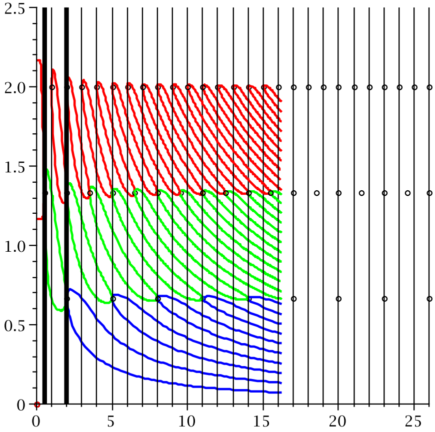
$$\vartheta_4^3(0|\tau)\imath\sqrt{3}\frac{\vartheta_1\left(\frac{\tau}{3}|\tau\right)\vartheta_4\left(\frac{\tau}{3}|\tau\right)}{\vartheta_2\left(\frac{\tau}{3}|\tau\right)^2}+\vartheta_4^2\left(0|\frac{\tau}{3}\right)\frac{\vartheta_1\left(\frac{1}{3}|\frac{\tau}{3}\right)\vartheta_4\left(\frac{1}{3}|\frac{\tau}{3}\right)}{\vartheta_3\left(\frac{1}{3}|\frac{\tau}{3}\right)^2}=0$$

We are able to prove that using Ramanujan third order transformation of Jacobian moduli

$$k(\tau)\equiv\frac{\vartheta_2(0|\tau)^2}{\vartheta_3(0|\tau)^2}=\frac{(p+1)^3(3-p)}{16p},$$

$$k(\tau/3)\equiv\frac{\vartheta_2(0|\tau/3)^2}{\vartheta_3(0|\tau/3)^2}=\frac{(p+1)(3-p)^3}{16p^3}$$

No charge 3 monopoles beside tetrahedral monopole



Three branches of the function y plotted against $(n+m)/(2n-m)$

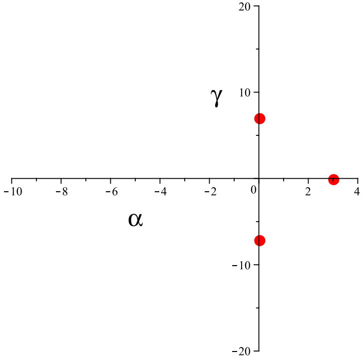
Only two cases $(n+m)/(2n-m) = 2$ and $(n+m)/(2n-m) = 1/2$ satisfy **H3**

Charge 3 monopole curve with cyclic symmetry

The genus four curve $\widehat{\mathcal{C}} = (\zeta, \eta)$ satisfying to **H1**

$$\eta^3 + \alpha\eta\zeta^2 + \zeta^6 + \gamma\zeta^3 - 1 = 0, \qquad \alpha, \gamma \in \mathbb{R}$$

But only 3 points were explicitly known:

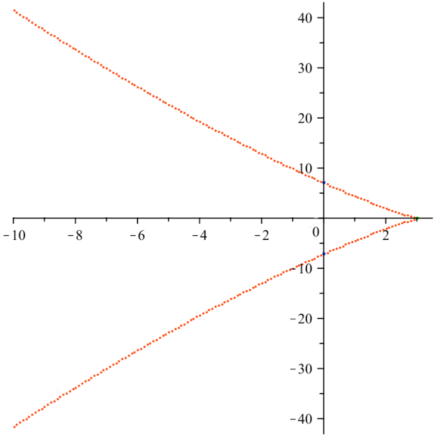


Do other points exist in the (α, γ)-plane?

New monopole curve (Braden,D'Avanzo&E, 2010)

The above result can be extended to the curve of genus 4

$$\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0$$



Axis α - horizontal and γ - vertical.

Richelot Arithmetic-Geometric mean, Bost-Mestre, 1988

Above genus four curve covers 3-sheetedly the genus two curve

$$y^2 = (x^3 + \alpha x + \gamma)^2 + 4\beta^2$$

and Schottky-Jung factorization is still applicable.
H2 is formulated as a condition on complete holomorphic integrals over this genus two curve:

$$\oint_{\mathfrak{c}} \frac{\mathrm{d}x}{y} = 0, \quad \oint_{\mathfrak{c}} \frac{x\mathrm{d}x}{y} = 6\beta^{1/3}$$

taken along certain cycle $\mathfrak{c}$.

Part II: Calculation of monopole fields

Monopole fields are given as

$$\Phi(\mathbf{x})_{\mu\nu} = \imath \int_{-1}^{1} z\mathbf{v}_\mu^\dagger(z,\mathbf{x}) \,.\, \mathbf{v}_\nu(z,\mathbf{x})\mathrm{d}z, \qquad \mu,\nu = 1,2, \quad i = 1,2,3,$$

$$a_i(\mathbf{x})_{\mu\nu} = \imath \int_{-1}^{1} \mathbf{v}_\mu^\dagger(z,\mathbf{x}) \,.\, \partial_{x_i}\mathbf{v}_\nu(z,\mathbf{x})\mathrm{d}z,$$

with **v** - solutions to the Weyl equation

$$\left(-\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}z} + \sum_{j=1}^{3}(T_j(s) + \imath x_j 1_n)\otimes\sigma_j\right)\mathbf{v}(z,\mathbf{x}) = 0$$

Panagopoulos formulae (1983)

Introduce

$$\mathcal{H}(\mathbf{x}) = \sum_{i=1}^{3} x_i \sigma_i \otimes 1_n, \quad \mathcal{T}(z) = \frac{1}{2}\sum_{k=1}^{3} T_k(z) \otimes \sigma_k$$

$$\mathcal{Q}(\mathbf{x}, z) = \frac{1}{r^2}\mathcal{H}(\mathbf{x})\mathcal{T}(z)\mathcal{H}(\mathbf{x}) - \mathcal{T}(z).$$

Then antiderivatives are computed as

$$\int \mathbf{v}_p^\dagger(z,\mathbf{x})\mathbf{v}_q(z,\mathbf{x})\mathrm{d}z = \mathbf{v}_p^\dagger(\mathbf{x}, z)\mathcal{Q}^{-1}(\mathbf{x}, z)\mathbf{v}_q(z,\mathbf{x})$$

$$\Phi_{p,q}(\mathbf{x}) \sim \int z\mathbf{v}_p^\dagger(z,\mathbf{x})\mathbf{v}_q(z,\mathbf{x})\mathrm{d}z$$

$$= \mathbf{v}_p^\dagger(z,\mathbf{x})\mathcal{Q}^{-1}(z,\mathbf{x})\left[z + \mathcal{H}(\mathbf{x})\sum_{i=1}^{3}\frac{x_i}{r^2}\frac{\partial}{\partial x_i}\right]\mathbf{v}_q(z,\mathbf{x});$$

Note We need only values of $\mathbf{v}_q(z,\mathbf{x})$ at boundaries of the interval,

$$\mathbf{v}_q(\pm 1, \mathbf{x}), \quad q = 1, 2$$

Similar formula for gauges a_i

$$
\begin{aligned}
a_{i;p,q}(\mathbf{x}) &\sim \int \mathbf{v}_p^\dagger(z,\mathbf{x}) \frac{\partial}{\partial x_i} \mathbf{v}_q(z,\mathbf{x}) \mathrm{d}z \\
&= \mathbf{v}_p^\dagger(z,\mathbf{x}) \mathcal{Q}^{-1}(z,\mathbf{x}) \left[\frac{\partial}{\partial x_i} + \mathcal{H}(\mathbf{x}) \frac{z x_i + \imath(\mathbf{x} \times \nabla)_i}{r^2} \right] \mathbf{v}_q(z,\mathbf{x}).
\end{aligned}
$$

Lesser known Nahm Ansatz I; $n \geq 2$, Nahm (1982)

Apart from the **Weyl equation**, $\Delta^\dagger \mathbf{v} = 0$

$$\left(\imath 1_{2n} \frac{\mathrm{d}}{\mathrm{d}z} - \sum_{j=1}^{3} (T_j + \imath x_j 1_n) \otimes \sigma_j \right) \begin{pmatrix} v_1(\mathbf{x}, z) \\ \vdots \\ v_{2n}(\mathbf{x}, z) \end{pmatrix} = 0$$

introduce **construction equation** $\Delta \mathbf{w} = 0$

$$\left(\imath 1_{2n} \frac{\mathrm{d}}{\mathrm{d}z} + \sum_{j=1}^{3} (T_j + \imath x_j 1_n) \otimes \sigma_j \right) \begin{pmatrix} w_1(\mathbf{x}, z) \\ \vdots \\ w_{2n}(\mathbf{x}, z) \end{pmatrix} = 0$$

If

$$W = (\mathbf{w}_1, \ldots, \mathbf{w}_{2n})$$

be fundamental solution to $\Delta \mathbf{w} = 0$, then fundamental solution

$$V = (\mathbf{v}_1, \ldots, \mathbf{v}_{2n})$$

to $\Delta^\dagger \mathbf{v} = 0$ reads

$$V = W^{-1\dagger}$$

Lesser known Nahm Ansatz II

W.Nahm introduced Ansatz to solve $\Delta \mathbf{w} = 0$

$$\mathbf{w}(\mathbf{x}, z) = (1_2 + \sum_{j=1}^{3} u_j(\zeta)\sigma_j)\boldsymbol{\chi} \otimes \boldsymbol{\psi}(z, \zeta), \quad \zeta = \zeta(\mathbf{x})$$

Here ζ-certain parameter, $\mathbf{u}(\zeta)$ real unit vector independent in z

$$\mathbf{u} = (u_1, u_2, u_3), \quad u_1^2 + u_2^2 + u_3^2 = 1$$

is constructed in terms of vector $\mathbf{y}$

$$\mathbf{y} = \left(\frac{1+\zeta^2}{2\imath}, \frac{1-\zeta^2}{2\imath}. -\zeta \right), \quad \mathbf{y} \cdot \mathbf{y} = 0$$

$$\mathbf{u} = \imath \frac{\mathbf{y} \times \mathbf{y}}{\mathbf{y} \cdot \overline{\mathbf{y}}}$$

$\boldsymbol{\psi}(z, \zeta)$ - n-vector to be found, also $\boldsymbol{\chi}$ arbitrary constant n- vector.

Then the construction equation reduces to the spectral problem,

$$L(z,\zeta)\psi(z,\zeta) = \eta\,\psi(z,\zeta)$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + M(z,\zeta)\right)\psi(z,\zeta) = 0$$

L, M are exactly $n \times n$-matrices of the Lax form of Nahm eqns.
The spectral curve

$$\eta^n + \alpha_1(\zeta)\eta^{n-1} + \ldots + \alpha_n(\zeta) = 0$$

is constraint by the condition

$$\eta = (x_2 + \imath x_1)\zeta^2 + 2x_3\zeta + x_2 - \imath x_1 \equiv \mathcal{P}_2$$

That is algebraic equation of order $2n$ with respect to ζ, called **Atiyah-Ward constraint**

$$\mathfrak{P}_{2n}(\zeta) = \mathcal{P}_2^n + \alpha_1(\zeta)\mathcal{P}_2^{n-1}\ldots + \alpha_n(\zeta) = 0$$

Resume on the "Lesser known Nahm Ansatz"

- Let $\mathcal{C}$ be monopole curve of genus $(n-1)^2$
- Let $\mathfrak{P}_{2n}(\zeta)$ be $2n$ degree polynomial vanishing in $2n$ points

$$\zeta_k(\mathbf{x}), \quad k = 1, \dots, 2n$$

- Let $\psi(z, \zeta(\mathbf{x}))$ be n-dimensional vector resolving of the linear problem in the Lax representation of Nahm equation
- Let $\mathbf{w}(z, \zeta(\mathbf{x}))$ be $2n$-dimensional vector described above
- Let $W = (\mathbf{w}(z, \zeta_1(\mathbf{x})), \dots, \mathbf{w}(z, \zeta_{2n}(\mathbf{x})))$ be $2n \times 2n$ matrix representing fundamental solution to the construction equation.

- Then fundamental solution V to the Weyl equation is given as

$$V = {W^{\dagger}}^{-1}$$

Nahm Anzatz revisited I

(**i**) Linear problem in Lax representation is non-standard and reads

$$\frac{\mathrm{d}\boldsymbol{\psi}}{\mathrm{d}z} + \frac{1}{2}A_0(z)\boldsymbol{\psi} = \zeta \cdot A_1(z)\boldsymbol{\psi}$$

$$A_1(s) = T_1(z) + \imath T_2(z), \quad A_0(z) = 2\imath T_3(z)$$

Gauge transform, $\mathcal{G}$

$$\boldsymbol{\psi}(z,\zeta) = \mathcal{G}\boldsymbol{\Phi}(z,\zeta)$$

should be done to reduce the spectral problem to standard form

$$\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}z} + Q(z)\boldsymbol{\Phi} = \zeta \cdot \mathrm{Diag}(1, \rho, \rho^2, \ldots, \rho^{n-1})\boldsymbol{\Phi}$$

Gauge transform I

Introduce

$$h = \mathcal{G}^{\dagger}\mathcal{G}$$

Recently (**Braden&E, CMP, 2018, in press**) found

$$h = \widehat{\Phi}(z, \mathbf{0})\widehat{\Phi}(0, \mathbf{0})^{-1}$$

with matrix Baker-Akhiezer function

$$\widehat{\Phi}(z, \zeta) = (\boldsymbol{\Phi}_1(z, P_1), \ldots, \boldsymbol{\Phi}_n(z, P_n)), \quad P_j = (\zeta, \eta_j)$$

$\widehat{\Phi}(0, \mathbf{0})$ - special values of θ-functions

Gauge transform II

Recall: to compute monopole fields via Panagopoulos formulae we need the quantity containing Nahm data $T_k(z)$,

$$\mathcal{T}(z) = \frac{1}{2}\sum_{k=1}^{3} T_k(z) \otimes \sigma_k$$

For this purpose we found (**Braden&E, CMP, 2018, in press**)

$$\mathcal{T}(z) = \begin{pmatrix} \frac{1}{2}\dot{h}h^{-1} & -\imath\nu^\dagger \\ \imath h\nu h^{-1} & -\frac{1}{2}\dot{h}h^{-1} \end{pmatrix}, \quad \dot{h} = \frac{\mathrm{d}h}{\mathrm{d}z}$$

Here

$$\nu = T_1(0) + \imath T_2(0) = \mathrm{Diag}(\nu_1, \ldots, \nu_n)$$

For calculation of monopole fields we need only h

Nahm Anzatz revisited II

(**ii**) To find expansion of matrix $V(z)$ near $z = 1 - \xi$, $z = -1 + \xi$ up to required order by the expansion $W(z)$ near $z = \pm 1$, $VW^{\dagger} = 1$

(**iii**) To find projection to 2-dimensional subspace of normalized vectors

(**ii**) and (**iii**) overcame at the case $n = 2$

Part III: charge two monopole

Hitchin constraints **H1.**,**H2.**,**H3.** constrain nothing in this case.
The curve:

$$\eta^2 + \frac{K^2}{4}(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1) = 0$$

The Atiyah-Ward constraint:

$$\left[(x_2 + \imath x_1)\zeta^2 + 2x_3\zeta + x_2 - \imath x_1\right]^2 = \frac{K^2}{4}\left(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1\right)$$

Nahm equation in this case resolved in Jacobi elliptic functions

$$T_j(z) = -\frac{1}{2}\sigma_j f_j(z), \quad j = 1, 2, 3$$

$$f_1(z) = K\frac{\mathrm{dn}(Kz;k)}{\mathrm{cn}(Kz;k)}, \quad f_2(z) = Kk'\frac{\mathrm{sn}(Kz;k)}{\mathrm{cn}(Kz;k)}, \quad f_3(z) = Kk'\frac{1}{\mathrm{cn}(Kz;k)}$$

Expansions of $\mathbf{v}$

Typical entry to the Panagopoulos formulae is of the form

$$\mathbf{v}^\dagger(z, \zeta_i(\mathbf{x}))\mathcal{Q}^{-1}(\mathbf{x}, z)\mathbf{v}(z, \zeta_i(\mathbf{x}))$$

$\mathcal{Q}$-matrix expands near $z = \pm 1$ as

$$\mathcal{Q}^{-1}(1-\xi) = \frac{\mathcal{Q}_+^{-1}}{\xi} + O(1), \quad \mathcal{Q}^{-1}(-1+\xi) = \frac{\mathcal{Q}_-^{-1}}{\xi} + O(1)$$

We need terms of order $\xi^{1/2}$ to find monopole fields.

$$\mathbf{v}(1-\xi, \zeta_i(\mathbf{x})) = \frac{1}{\xi^{3/2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{\xi^{1/2}}\begin{pmatrix} \imath x_2 - x_1 \\ x_3 \\ x_3 \\ \imath x_2 + x_1 \end{pmatrix} + \xi^{1/2}\begin{pmatrix} a_i(\mathbf{x}) \\ b_i(\mathbf{x}) - r^2/2 \\ b_i(\mathbf{x}) + r^2/2 \\ c_i(\mathbf{x}) \end{pmatrix} + \dots$$

Expansions $\mathbf{v}$ near $z = \pm 1 \mp \xi$ showing monodromy

$$\mathbf{v}(1-\xi,\zeta(\mathbf{x}))$$

$$= \frac{1}{\xi^{3/2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{\xi^{1/2}} \begin{pmatrix} \imath x_2 - x_1 \\ x_3 \\ x_3 \\ \imath x_2 + x_1 \end{pmatrix} + \xi^{1/2} \begin{pmatrix} a \\ b - r^2/2 \\ b + r^2/2 \\ c \end{pmatrix} + O(\xi^{3/2})$$

$$\mathbf{v}(-1+\xi,\zeta(\mathbf{x}))$$

$$= \frac{1}{\xi^{3/2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\xi^{1/2}} \begin{pmatrix} -x_3 \\ \imath x_2 - x_1 \\ -\imath x_2 + x_1 \\ x_3 \end{pmatrix} + \xi^{1/2} \begin{pmatrix} a' - r^2/2 \\ b' \\ c' \\ -a' - r^2/2 \end{pmatrix} + O(\xi^{3/2})$$

We should find $a, b, c, a' b' c'$ from the relation

$$V = W^{-1\dagger}$$

Fundamental solution of $\Delta \mathbf{w} = 0$

Columns $\mathbf{w}_k$, of the fundamental solution $W = (\mathbf{w}_1, \ldots, \mathbf{w}_4)$ are

$$\mathbf{w}_k = \begin{pmatrix} 1 \\ \imath\zeta_k \end{pmatrix} \otimes \begin{pmatrix} -\vartheta_3(\alpha_k)\vartheta_2(\alpha_k - z/2) \\ \\ \vartheta_1(\alpha_k)\vartheta_4(\alpha_k - z/2) \end{pmatrix} \frac{\exp\{\beta_k z\}}{\vartheta_2\,(z/2)}$$

$$\text{Here} \quad \alpha_k = \int_{\infty}^{\zeta_k} \omega, \quad \beta_k = \int_{\zeta_0}^{\zeta_k} \gamma_\infty, \quad k = 1, \ldots, 4$$

ω and γ_∞ - first and second kind normalised differentials, ζ_0 is a branch point.

$$\mathrm{Det}\, W = \frac{\prod_{i<j} \vartheta_1(\alpha_i - \alpha_j)}{\prod_{k=1}^{4} \vartheta_1(\alpha_k)\vartheta_3(\alpha_k)} \exp\left\{-\imath\pi(N^2\tau - Nz)\right\}, \quad N \in \mathbb{Z}$$

Determinant $\mathrm{Det}\, W$ computed using the **Weierstrass trisecants**

Weierstrass trisecant relations (Weierstrass-Schwartz Lectures (1885)

Let $\boldsymbol{\alpha}' = T(\boldsymbol{\alpha})$, $\boldsymbol{\alpha}'' = T(\boldsymbol{\alpha}')$, $T(\boldsymbol{\alpha}'') = \boldsymbol{\alpha}$ and

$$T\,.\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\alpha_1+\alpha_2+\alpha_3+\alpha_4\\ \alpha_1+\alpha_2-\alpha_3-\alpha_4\\ \alpha_1-\alpha_2+\alpha_3-\alpha_4\\ -\alpha_1+\alpha_2+\alpha_3-\alpha_4\end{pmatrix}$$

Weierstrass-Schwartz gave 6 trisecant formulae, W1,.., W6, we present here those two which we used

$$[W1]\quad \vartheta_1(\alpha_1)\vartheta_1(\alpha_2)\vartheta_1(\alpha_3)\vartheta_1(\alpha_4)+\vartheta_1(\alpha_1')\vartheta_1(\alpha_2')\vartheta_1(\alpha_3')\vartheta_1(\alpha_4') + \vartheta_1(\alpha_1'')\vartheta_1(\alpha_2'')\vartheta_1(\alpha_3'')\vartheta_1(\alpha_4'')=0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$[W6]\quad \vartheta_i(\alpha_1)\vartheta_i(\alpha_2)\vartheta_i(\alpha_3)\vartheta_i(\alpha_4)-\vartheta_i(\alpha_1')\vartheta_i(\alpha_2')\vartheta_i(\alpha_3')\vartheta_i(\alpha_4') \pm \vartheta_1(\alpha_1'')\vartheta_1(\alpha_2'')\vartheta_1(\alpha_3'')\vartheta_1(\alpha_4'')=0$$

Expansions of W and V

Let $W(z, \mathbf{x})$, $V(z, \mathbf{x})$ -fundamental solutions of $\Delta^\dagger V = 0$, $\Delta W = 0$, then

$$W(1-\xi, \mathbf{x}) = \frac{1}{\xi^{1/2}} W_0 + \xi^{1/2} W_1 + \xi^{3/2} W_2 + O(\xi^{5/2})$$
$$V(1-\xi, \mathbf{x}) = \frac{1}{\xi^{3/2}} V_0 + \frac{1}{\xi^{1/2}} V_1 + \xi^{1/2} V_2 + O(\xi^{3/2})$$

Using Nahm condition $VW^\dagger = 1$ reduces to

$$W_0^T . V_2 + W_1^T . V_1 + W_2^T . V_0 = 0$$

compute V_2 via Kramer rule.

Energy density $\mathcal{E}(\mathbf{x})$, $\quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\mathcal{E}(\mathbf{x}) = \mathrm{Trace}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\left(\Phi . G^{-1}, \Phi . G^{-1}\right)$$

with

$$\begin{aligned} \Phi(\mathbf{x})_{\mu\nu} &= \imath \int_{-1}^{1} z \mathbf{v}_\mu^\dagger(z, \mathbf{x}) \,.\, \mathbf{v}_\nu(z, \mathbf{x}) \mathrm{d}z, \\ G(\mathbf{x})_{\mu\nu} &= \int_{-1}^{1} \mathbf{v}_\mu^\dagger(z, \mathbf{x}) \,.\, \mathbf{v}_\nu(z, \mathbf{x}) \mathrm{d}z, \end{aligned} \qquad \mu, \nu = 1, 2$$

Energy density $\mathcal{E}(\mathbf{x})$ for $n = 2$

Fix elliptic curve

$$\mathcal{C} = (\zeta, \eta) : \qquad \eta^2 + \frac{K^2}{4}(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1) = 0$$

Find four solutions

$$\zeta_j(\mathbf{x}), \qquad j = 1, \dots 4$$

of quartic equation, **Atiyah-Ward constraint**

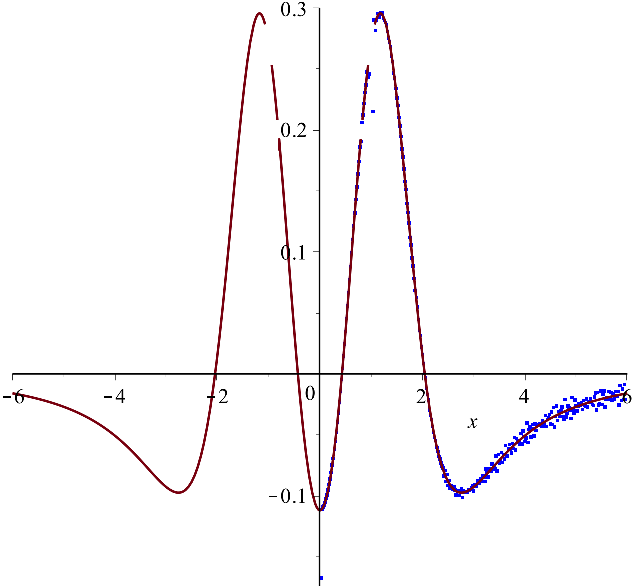
$$\eta = (x_2 + \imath x_1)\zeta^2 + 2x_3\zeta + x_2 - \imath x_1$$

Find four transcendents

$$\mu_j(\mathbf{x}) = \exp\left\{\int_{k'+\imath k}^{\zeta_j(\mathbf{x})} \frac{\mathrm{d}\zeta}{\eta}\left(\zeta^2 - \frac{2E-K}{K}\right)\right\}, \quad j = 1, \dots, 4$$

Energy density $\mathcal{E}(\mathbf{x})$ is expressible in terms of ζ_j, μ_j and the above formula.

Numerics by P.Sutcliffe: Energy density along x_1 axis



Description of monopole curve satisfying **H2.** and **H3.**

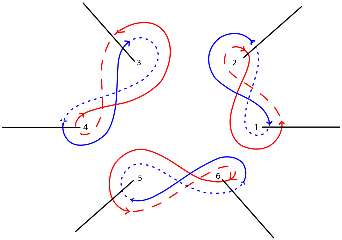
$$\mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_n}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m},$$

$$\theta(\mathbf{U}s + \mathbf{K};\tau) \neq 0, \quad s \in (0,2)$$

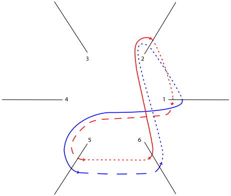
Algebro-geometric approach predicts $4n-4$ dimension of the monopole moduli space; 3 parameters - coordinates of the centrum, i.e. $4n-7$. At $n=2$ - one parametr, Jacobi k
At $n=3$ - 5 parametrs. The method exposed permits to find the point associated to the Plato solid - tetrahedron and then find one-dimensional subspace in $5D$ space of parametrs.

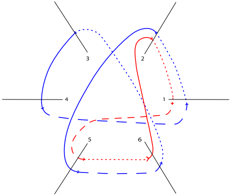
Homologies evaluation for curves with symmetries

Easy to construct first 3 cycles



and difficult to find the fourth





Right plot admits symmetries needed for Fay-Accola theorem
Problem: construct homologies respecting symmeries of the curve

Surprising θ-relations I

Given:

Elliptic curve $\eta^2 + \frac{K^2}{4}\left(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1\right) = 0$

Atiyah-Ward constraint $\eta = (x_2 + \imath x_1)\zeta^2 + 2x_3\zeta + x_2 - \imath x_1$

Abelian images $\alpha_i = \int^{\zeta_i} \omega, \quad \alpha_1 + \ldots + \alpha_4 = N\tau, \quad N \in \mathbb{Z}$

Second kind integrals $\beta_i = \int_{\zeta_0}^{\zeta_i} \gamma_\infty, \quad \beta_1 + \ldots + \beta_4 = -\frac{\imath\pi}{2}$

$$\vartheta_1(\alpha_i + \alpha_j + \alpha_k) \quad \text{at} \quad i \neq j \neq k$$
$$= \frac{(2x_1 - 2\imath x_2 - K)\vartheta_3(\alpha_i + \alpha_j)\vartheta_3(\alpha_j + \alpha_k)\vartheta_3(\alpha_k + \alpha_i)}{\pi\vartheta_3(0)\vartheta_1(\alpha_i)\vartheta_1(\alpha_j)\vartheta_1(\alpha_k)}$$

$$\vartheta_3(\alpha_i + \alpha_j + \alpha_k)$$
$$= \frac{(2x_1 - 2\imath x_2 + K)\vartheta_3(\alpha_i + \alpha_j)\vartheta_3(\alpha_j + \alpha_k)\vartheta_3(\alpha_k + \alpha_i)}{\pi\vartheta_3(0)\vartheta_3(\alpha_i)\vartheta_3(\alpha_j)\vartheta_3(\alpha_k)}$$

Surprising θ-relations II

At $i \neq j \neq k \neq l \in \{1, \ldots, 4\}$

$$\frac{\vartheta_3'(\alpha_i + \alpha_j + \alpha_k)}{\vartheta_3(\alpha_i + \alpha_j + \alpha_k)} = -2\beta_l + \imath K \zeta_l - 2\imath\pi N$$

$$\frac{\vartheta_1'(\alpha_i + \alpha_j + \alpha_k)}{\vartheta_1(\alpha_i + \alpha_j + \alpha_k)} = -2\beta_l - \imath K \zeta_l - 2\imath\pi N$$

At $i \neq j \neq k \neq l \in \{1, \ldots, 4\}$

$$\frac{\vartheta_3''(\alpha_i + \alpha_j)}{\vartheta_3(\alpha_i + \alpha_j)} = -2(\beta_k + \beta_l) + 4x_3 + 2(x_2 + \imath x_1)(\zeta_k + \zeta_l) - 2\imath\pi N$$

and others following from the above given.

Problem: Generalize above relations to higher charges, $n > 2$ and monopole curves.

Theta-constant representation of periods, $g = 1$

Jacobi

$$K = \frac{\pi}{2}\vartheta_3^2(0;\tau), \quad \tau = \imath\frac{K'}{K}$$

$$K = K(k) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

Weierstrass

$$\eta = -\frac{1}{12\omega}\left(\frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}\right)$$

$$2\omega = \oint_{\mathfrak{a}} \frac{\mathrm{d}x}{y}, \quad 2\eta = -\oint_{\mathfrak{a}} \frac{x\mathrm{d}x}{y}$$
$$2\omega' = \oint_{\mathfrak{b}} \frac{\mathrm{d}x}{y}, \quad 2\eta' = -\oint_{\mathfrak{b}} \frac{x\mathrm{d}x}{y}$$

$$\tau = \frac{\omega'}{\omega}$$

Legendre relation $\omega\eta' - \eta\omega' = -\frac{\imath\pi}{2}$

First kind periods at $g = 2$ (Rosenhain, 1851)

$$y^2 = x(x-1)(x-a_1)(x-a_2)(x-a_3)$$

and matrix $\mathcal{A}$ of $\mathfrak{a}$-periods,

$$(2\omega)^{-1} = \left(\oint_{\mathfrak{a}_j} \frac{x^{i-1}\mathrm{d}x}{y} \right)^{-1}_{i,j=1,2} = \frac{1}{2\pi^2 Q^2} \begin{pmatrix} -P\theta_2[\delta_2] & Q\theta_2[\delta_1] \\ P\theta_1[\delta_2] & -Q\theta_1[\delta_1] \end{pmatrix}$$

with

$$P = \theta[\alpha_1]\theta[\alpha_2]\theta[\alpha_3], \quad Q = \theta[\beta_1]\theta[\beta_2]\theta[\beta_3]$$

$$\theta_i[\delta] = \frac{\partial}{\partial z_i}\theta[\delta](z_1, z_2; \tau)\Big|_{\boldsymbol{z}=0}, \quad i = 1, 2, \quad [\delta] \text{ odd}$$

and 6 even characteristics $[\alpha_{1,2,3}], [\beta_{1,2,3}]$ and two odd $[\delta_{1,2}]$

Superstructure of Rosenhain derivative formula

Take any of 15 Rosenhain derivative formulas,

$$\theta_1[p]\theta_2[q] - \theta_2[p]\theta_1[q] = \pi^2\theta[\gamma_1]\theta[\gamma_2]\theta[\gamma_3]\theta[\gamma_4]$$

10 even characteristics can be grouped as

$$\underbrace{[\gamma_1],\ldots,[\gamma_4]}_{4},\ \underbrace{[\alpha_1],[\alpha_2],[\alpha_3]}_{[\alpha_1]+[\alpha_2]+[\alpha_3]=[p]},\ \underbrace{[\beta_1],[\beta_2],[\beta_3]}_{[\beta_1]+[\beta_2]+[\beta_3]=[q]},$$

Then

$$2\omega = \frac{2Q}{PR}\begin{pmatrix} Q\theta_1[q] & Q\theta_2[q] \\ P\theta_1[p] & P\theta_2[p] \end{pmatrix}$$

with

$$P = \prod_{j=1}^{3}\theta[\alpha_j],\quad Q = \prod_{j=1}^{3}\theta[\beta_j],\quad R = \prod_{j=1}^{4}\theta[\gamma_j]$$

Baker's basis of co-homologies, 1898, 1907

$$y^2 = 4x^5 + \lambda_4 x^4 \ldots + \lambda_0$$

$$u_i = \frac{x^{i-1}}{y}\mathrm{d}x, \quad i = 1, \ldots, g$$

$$r_j = \frac{1}{4y}\sum_{k=j}^{2g+1-j}(k+1-j)\lambda_{k+j+1}x^k\mathrm{d}x, \quad j = 1, \ldots, g$$

$$2\omega = \left(\oint_{\mathfrak{a}_j} u_i\right)_{i,j=1,\ldots,g}, \quad 2\eta = -\left(\oint_{\mathfrak{a}_j} r_i\right)_{i,j=1,\ldots,g}$$

$$2\omega' = \left(\oint_{\mathfrak{b}_j} u_i\right)_{i,j=1,\ldots,g}, \quad 2\eta' = -\left(\oint_{\mathfrak{b}_j} r_i\right)_{i,j=1,\ldots,g}$$

This basis satisfies to the **Generalized Legendre relation**

$$\eta'\omega^T - \eta\omega'^T = -\frac{1}{2}\imath\pi, \quad \omega'\omega^T - \omega\omega'^T = 0 \quad \eta'\eta^T - \eta\eta'^T = 0$$

Second kind periods (Klein, 1888)

E&Eilbeck,Eilers, 2013

$$\eta(2\omega)^{-1} = \frac{1}{8}\frac{1}{10}\begin{pmatrix} 4\lambda_2 & \lambda_3 \\ \lambda_3 & 4\lambda_4 \end{pmatrix} - \frac{1}{2}\frac{1}{10}\sum_{10\,[\varepsilon]} \frac{\begin{pmatrix} \Theta_{11}[\varepsilon] & \Theta_{12}[\varepsilon] \\ \Theta_{12}[\varepsilon] & \Theta_{22}[\varepsilon] \end{pmatrix}}{\theta[\varepsilon]}$$

Directional derivatives $\Theta_{i,j}[\varepsilon] = \partial_{\mathbf{U}_i}\partial_{\mathbf{U}_j}\theta[\varepsilon], \quad (2\omega)^{-1} = (\mathbf{U}_1, \mathbf{U}_2)$

Generalization of these formulae to hyperelliptic curves in **K.Eilers, 2016, 2018**

Non-hyperelliptic curves are not studies in this context