# THE SHAFAREVICH CONJECTURE ABOUT ABELIAN VARIETIES /$\mathbb{Q}$ WITH EVERYWHERE GOOD REDUCTION AND RELATED TOPICS

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## Introduction

We discuss the following results.

1. There are no abelian varieties over $\mathbb{Q}$ with everywhere good reduction.

2. If $X$ is a projective smooth algebraic variety with everywhere good reduction then for its Hodge numbers we have $h^1 = 0$, $h^2 = h^{1,1}$, $h^3 = 0$ (and $h^4 = h^{2,2}$ modulo General Riemann Hypothesis).

3. If $X/\mathbb{Q}$ is projective smooth with bad semi-stable reduction modulo 3 and good reduction modulo all primes $l \neq 3$ then $h^2 = h^{1,1}$.

We give more detailed exposition of some parts of the author overview "A semi-stable case of the Shafarevich Conjecture" (to appear in "Automorphic forms and galois representations", London Mathematical Society lecture note series, v. 414, pp.1-31, cf. also the online version on the site "www.maths.dur.ac.uk/ dma0va/"). All references will be given below according to the list of references from that paper.

The first result appears as an application of the theory of finite flat group schemes over $\mathbb{Z}$. The main reference is Fontaine's paper [21], but we follow alternative author's approach from [3,4,5] paying attention also to earlier results from [1,2].

The second result appears as an application of Fontaine's theory of crystalline representations, the Fontaine-Messing result on the comparison of etale and De Rham cohomology in dimensions $< p$ (which was proved later in a full generality by Faltings), and the Fontaine-Laffaille theory. The main reference is Fontaine's paper [23] and slightly alternative author's approach (together with the modification of the Fontaine-Laffaille theory to the case of Hodge-Tate weights from $[0, p-1]$) from [6,7].

The third result appears as an application of the theory of semi-stable representations and in particular of the analogue of the Fontaine-Laffaille theory of semistable torsion representations developed by Breuil, cf. [14-16]. Strictly speaking we need a modification of Breuil's theory to the case of the HT weights from $[0, p-1]$ developed in the paper [11], which contains also a complete proof of this (third) result.

Note that in the first two cases we have also similar results for varieties defined over some algebraic number fields with small discriminants and the appropriate methods have been considerably developed in 2000's mainly in two directions:

— abelian varieties with everywhere good reduction, cf. Schoof's paper in Math Annalen (2003);

— Abelian varieties with bad semi-stable reduction at one small prime $p$ and good reduction at all primes $l \neq p$, cf. e.g. the results of Brumer-Cramer [13], Schoof [27], Verhoek (Journal Number Theory, 2013).

## 1. Abelian varieties over $\mathbb{Q}$ with everywhere good reduction

### 1.1. **ICM 1962, Stockholm.**

I.R.Shafarevich's talk "Algebraic Number Fields":

— $[K:\mathbb{Q}] < \infty$ $\leftrightarrow$ Riemannian surfaces over $\mathbb{C}$

— $S \subset \mathrm{Div}K$ finite, the Galois groups $\Gamma_{K,S}$, $\Gamma_{K,S}(p)$

$\leftrightarrow$ ramification of maps of Riemannian surfaces

(the problem of existence of infinite Hilbert's towers if $S = \emptyset$!)

— Galois groups of $p$-ext local fields $\leftrightarrow$ Burnside problem & fundamental groups (Demushkin)

— $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) =?$ (later done by Koch, Jakovlev, Jannssen-Wingberg)

Analogs to Algebraic Number Theory:

• $K'/K$ alg.n.f., critical ($\equiv$ unramified) $S \subset \mathrm{Div}K$ requires integral models $O_{K'}/O_K$

— $\{K' \mid [K':K] = const, S(K'/K) = const\}$ is finite (**Hermite**);

— $\{K' \mid S(K'/\mathbb{Q}) = \emptyset\} = \{\mathbb{Q}\}$ (**Minkowski**).

• $X/K$ proper non-singular algebraic variety;

integral model $\mathcal{X}/O_K$ $\to$ divisors of bad reduction $S(X) \subset \mathrm{Div}K$,

$|S(X)| < \infty$ $\to$ simplest case $\dim_K X = 1$, $g = g(X)$ — genus:

**Conjectures**:

— $g > 1$ then $\{X/K \mid S(X) = S_0, g(X) = g\}$ is finite

— if $g = 1$ & $X(K) \neq \emptyset$ then {SIMILAR SET} is FINITE

— if $g \geqslant 1$ & $\{K = \mathbb{Q}\}$ & $\{S_0 = \emptyset\}$ then NO SUCH CURVES

All these conjectures have geometrical flavour because of the non-uniqueness of integral models.

**Example**:

• $g = 1$, $S(X) = \{2\}$ there are 24 CURVES:

— $y^2 = x(x^2 - a)$ with $a = \pm 1, \pm 2, \pm 4, \pm 8$;

— $y^2 = x(x - a\alpha)(x - a\alpha')$ with

$$a = \pm 1, \pm 2, \quad \alpha = 1 + i, 1 + \sqrt{2}, (1 + \sqrt{2})^2, 2 + \sqrt{2}$$

MAIN STEPS:

— $X$ : $y^2 = P(x)$ with cubic $P(X) \in \mathbb{Z}[X]$ — MINIMAL MODEL outside (2)

(by Riemann-Roch theorem $\mathbb{Q} \cdot 1 + \mathbb{Q}x + \mathbb{Q}y$ is well-defined in $\mathbb{Q}(X)$)

— GOOD REDUCTION away from (2) $\equiv$ $\{Disc(P) = \text{POWER OF } 2\}$

— $P(X)$ is REDUCIBLE (use class field theory!)

— $P(X) = X(X - \alpha)(X - \beta)$ where $\alpha$, $\beta$, $\alpha - \beta$ are coprime to (2) alg. integers

• $g = 1$, $S = \emptyset$ $\equiv$ all above 24 curves have bad reduction at 2

Literally, Shafarevich stated:

• *Is there a field of algebraic functions in one variable over $\mathbb{Q}$ without critical prime numbers different from $\mathbb{Q}(t)$?*

In modern language:

• *There are no projective algebraic curves over $\mathbb{Q}$ with genus $\geq 1$ and everywhere good reduction.*

Later his PhD student (Volynsky) studied curves of GENUS 2 ($\equiv$ hyperelliptic!); enormous calculations (not published)

Later switching: $\{curves\} \mapsto \{AV's\}$

• *There are no abelian varieties over $\mathbb{Q}$ of dimension $g \geq 1$ with everywhere good reduction.*

This is more general:

— $X \mapsto \mathrm{Jac}X$ has polarization of degree 1;

— $\mathrm{Jac}X \bmod l$ can be non-singular for singular $X \bmod l$;

It is easier to approach: AV has ADDITIONAL STRUCTURE

Finally, it can be stated as

• *There are no non-trivial abelian schemes over $\mathbb{Z}$.*

Affine group schemes over $\mathbb{Z}$ do exist — e.g. $\mathbb{G}_m$, $\mathbb{G}_a$;

Highly delicate geometrical property:

*if $\mathcal{A}$ is abelian scheme $/\mathbb{Z}$ then $N\mathrm{id}_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ is faithfully flat.*

If such $\mathcal{A}$ exists then $\mathrm{Ker}\,(N\mathrm{id}_{\mathcal{A}})$ is a finite flat group scheme over $\mathbb{Z}$ of order $N^{2g}$.

Problem (Geometry has gone!):

— What are FINITE, FLAT GROUP SCHEMES (ffgs) over $\mathbb{Z}$?

## 1.2. ffgs over a ring $R$ (quick reminder).

General definition over arbitrary (commutative with 1 etc) ring $R$;

$\mathrm{Alg}_R$ — FLAT $R$-algebras $A$, $\mathrm{rk}_R A < \infty$;

$\mathrm{Gr}_R$ — the category of $G = \mathrm{Spec} A$ with COALGEBRA structure

— COADDITION $\Delta : A \longrightarrow A \otimes A$

— COUNIT $e : A \longrightarrow R$;

— COINVERSION $i : A \longrightarrow A$

$|G| = \mathrm{rk}_R A$ — the ORDER of $G$.

Axioms are dual w.r.t. $A \mapsto \mathrm{Hom}_{R-mod}(A, R) \quad \longrightarrow$

— the CARTIER DUALITY $G \mapsto G^D$

$\mathrm{Gr}_R$ — additive category with Kernels, Cokernels;

$0 \longrightarrow G_1 \xrightarrow{i} G \xrightarrow{j} G_2 \longrightarrow 0$ is SHORT EXACT means:

— $i$ — closed embedding; $j$ — faithfully flat;

— $\mathrm{Ker}\, j = (G_1, i)$ & $\mathrm{Coker}\, i = (G_2, j)$

- $D(A(G)) = D(A(G_1))^{|G_2|} D(A(G_2))^{|G_1|}$.
- $\mathrm{Gr}_R$ is not abelian (but abelian if $R$ is a field).

## 1.3. Finite flat group schemes $/\mathbb{Z}$. :

**Relation to $\mathrm{Gr}_{\mathbb{Q}}$ and $\mathrm{Gr}_{\bar{\mathbb{Q}}}$**

— $G = \mathrm{Spec} A \in \mathrm{Gr}_{\mathbb{Z}}$ then

$A_{\mathbb{Q}} = \mathrm{Map}^{\Gamma_{\mathbb{Q}}}(G(\bar{\mathbb{Q}}), \bar{\mathbb{Q}}) = \mathbb{Q} \oplus (\bigoplus_{\alpha} K_\alpha)$, all $[K_\alpha : \mathbb{Q}] < \infty$.

— $H' \subset G \otimes \mathbb{Q}$ then $\exists! H \subset G$ s.t. $H \otimes Q = H'$

— $G \in \mathrm{Gr}_{\mathbb{Z}}$, $|G|$ is a power of $p$ then

all $K_\alpha/\mathbb{Q}$ are UNRAMIFIED away from $p$

**Constant etale $\epsilon_n$**

— $A(\epsilon_n) = \mathrm{Map}(\mathbb{Z}/n, \mathbb{Z})$ with

$e : A(\epsilon_n) \longrightarrow \mathbb{Z}$ such that $a \mapsto a(0)$;

$\Delta : A(\epsilon_n) \longrightarrow A(\epsilon_n) \otimes A(\epsilon_n)$ such that $\Delta(a)(h_1, h_2) = a(h_1 + h_2)$.

— Say, $A(\epsilon_2) = \mathbb{Z} \oplus \mathbb{Z}$; $A(\epsilon_{2,R}) = R \oplus R$.

**Remark.** One can define constant etale groups $H_R$ for any abstract (abelian) group $H$ over any ring $R$. The Minkowski Theorem implies that for $R = \mathbb{Z}$, any extension of constant etale via constant etale is again constant etale.

**Constant multiplicative**:

— $A(\mu_n) = \mathbb{Z}[\mathbb{Z}/n] = \mathbb{Z}[X]/(X^n - 1)$ with

$e : A(\mu_n) \longrightarrow \mathbb{Z}$ such that $X \mapsto 1$,

$\Delta : A(\mu_n) \longrightarrow A(\mu_n) \otimes A(\mu_n)$ such that $X \mapsto X \otimes X$.

— $A(\mu_2) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + b \equiv 0 \bmod 2\}$:

$1 \mapsto (1, 1)$, $X \mapsto (1, -1)$ gives

$\mathfrak{f}_0 : \varepsilon_2 \longrightarrow \mu_2$ such that $\mathfrak{f}_0^* : A(\mu_2) \subset \mathbb{Z} \oplus \mathbb{Z}$;

(!!) $\operatorname{Ker} \mathfrak{f}_0 = \operatorname{Coker} \mathfrak{f}_0 = 0$.

**For any prime $p$, $\varepsilon_p$ and $\mu_p$ — the only ffgs $/\mathbb{Z}$ of order $p$.** (Tate-Oort)

— $\mu_p^D = \epsilon_p$ and $\epsilon_p = \mu_p^D$ (Cartier duality)

**Group schemes $G$ of order 4 over $\mathbb{Z}$,** $2\mathrm{id}_G = 0$

— $\mathrm{Ext}_{2\mathrm{id}=0}(\epsilon_2, \epsilon_2) = 0$ — use Minkowski Thm;

— $\mathrm{Ext}_{2\mathrm{id}=0}(\mu_2, \mu_2) = 0$ (use Cartier duality);

— $\mathrm{Ext}_{2\mathrm{id}=0}(\mu_2, \varepsilon_2) = 0$ — use $D(A(G)) = 2^4$ & $G \otimes \mathbb{F}_2$ splits (max reduced subscheme) & $A(G) \subset \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$ (by Minkowski Thm);

— $\mathrm{Ext}_{2\mathrm{id}=0}(\varepsilon_2, \mu_2) = \mathbb{Z}/2$

$A(G_0) = A(\mu_2) \oplus B$ with $D(B/\mathbb{Z}) = 2^2$, i.e. $B = \mathbb{Z}[i]$; (exercise)

• $G \in \mathrm{Gr}_{\mathbb{Z}}$, $|G| = 4$, $2\mathrm{id}_G = 0$ THEN $G$ IS EITHER PRODUCT OF $\mu_2$ OR $\epsilon_2$, OR COINCIDES WITH $G_0$:

Indeed, two cases:

— $A(G) = \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots$ then one of the above;

— $A(G) = \mathbb{Q} \oplus K$, $[K : \mathbb{Q}] = 3$, ramified only at 2 $\Rightarrow$ no such $K$ (class field theory)

**Group schemes of order $p^2$, $p\mathrm{id}_G = 0$, $p > 2$**

— $\mathrm{Ext}_{p\mathrm{id}=0}(\epsilon_p, \epsilon_p) = \mathrm{Ext}_{p\mathrm{id}=0}(\mu_p, \mu_p) = 0$;

— $\mathrm{Ext}_{p\mathrm{id}=0}(\mu_p, \epsilon_p) = 0$ (property of Bernoulli numbers)

— $\mathrm{Ext}_{p\mathrm{id}=0}(\epsilon_p, \mu_p) = 0$ — special case that over any $R$, this group is $R^*/R^{*p}$:

$A(G) = \bigoplus_{i\in\mathbb{Z}/p} R[T_i]$, where $T_i^p = r \in R^*$ and $\mu_p$ acts via $T_i \mapsto x \otimes T_i$

($\equiv$ a description of principal homogeneous spaces of $\mu_p$ over ring $R$)

**Remark.** Any $G \in \mathrm{Gr}_{\mathbb{Z}}$, $|G| = p^2$, $p\mathrm{id}_G = 0$ is the product of $\mu_p$ or $\epsilon_p$ (corollary of Serre's conjecture proved by Khare and Wintenberger)

## 1.4. $p$-divisible groups $/R$.

$p$-DIVISIBLE GROUP in $\mathrm{Gr}_R$ is an inductive system $\mathcal{C} = (C^{(n)}, i_n)_{n\geqslant 0}$ of objects of $\mathrm{Gr}_R$ such that for all $0 \leqslant m \leqslant n$,

$$0 \longrightarrow C^{(m)} \xrightarrow{i_{mn}} C^{(n)} \xrightarrow{j_{n,n-m}} C^{(n-m)} \longrightarrow 0$$

$$\begin{array}{ccc} C^{(n)} & \xrightarrow{\quad p^m \mathrm{id}_{C^{(n)}} \quad} & C^{(n)} \\ & {\scriptstyle j_{n,n-m}} \searrow \qquad \nearrow {\scriptstyle i_{n-m,n}} & \\ & C^{(n-m)} & \end{array}$$

**Remark.** There is $h = h(\mathcal{C}) \in \mathbb{Z}$ (the HEIGHT of $\mathcal{C}$) s.t. for any $n$, $C^{(n)} = p^{nh}$.

**Examples** $/\mathbb{Z}$:

— CONSTANT ETALE $(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbb{Z}} = \{\epsilon_{p^n}\}_{n\geqslant 0}$

— CONSTANT MULTIPLICATIVE $(\mathbb{Q}_p/\mathbb{Z}_p)(1)_{\mathbb{Z}} = \{\mu_{p^n}\}_{n\geqslant 0}$

— TRIVIAL $p$-div group $\mathbb{Z}$ – the product of copies of constant etale and multiplicative

**Are there non-trivial $p$-divisible groups** $/\mathbb{Z}$? (J.TATE)

— $\mathcal{A}$ is abelian scheme over $\mathbb{Z}$ of dimension $g \geqslant 1$

$p$ a prime,

$p$-divisible group $\mathcal{A}(p)$ associated with $\mathcal{A}$.

$\mathcal{A}(p) = \varinjlim_n (\mathrm{Ker}(p^n \mathrm{id}_{\mathcal{A}})$ over $\mathbb{Z}$.

**No non-trivial $p$-div groups** $/\mathbb{Z} \overset{!}{\Longrightarrow}$ **Shafarevich Conjecture**.

Otherwise, $(\mathcal{A} \otimes \mathbb{F}_p)(\mathbb{F}_p)$ is infinite

1.5. **2-divisible groups of height $\leqslant 2$.**

$\mathcal{G} = \{G^{(n)}\}_{n\geqslant 0}$ — 2-div. group $/\mathbb{Z}$

— *if height of $\mathcal{G}$ is 1 then $\mathcal{G}$ is trivial*;

$G^{(1)} = \epsilon_2$ (use Minkowski),

$G^{(1)} = \mu_2$ (use Cartier duality)

— if $G^{(1)}$ is $\epsilon_2 \times \epsilon_2$ or $\mu_2 \times \mu_2$ then $G$ is trivial (similarly);

— $G^{(1)} \in \mathrm{Ext}(\epsilon_2, \mu_2)$, $\quad 0 \longrightarrow \mu_2 \xrightarrow{i_1} G^{(1)} \xrightarrow{j_1} \epsilon_2 \longrightarrow 0$

then for all $n$, $\quad 0 \longrightarrow \mu_{2^n} \xrightarrow{i_n} G^{(n)} \xrightarrow{j_n} \epsilon_{2^n} \longrightarrow 0$

and, therefore,

$$0 \longrightarrow (\mathbb{Q}_2/\mathbb{Z}_2)(1)_{\mathbb{Z}} \longrightarrow \mathcal{G} \longrightarrow (\mathbb{Q}_2/\mathbb{Z}_2)_{\mathbb{Z}} \longrightarrow 0$$

Prove this for $n = 2$.

— $i_1^*$ induces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G^{(1)} & \xrightarrow{i_{12}} & G^{(2)} & \xrightarrow{j_{21}} & G^{(1)} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle i_1} & & \\
0 & \longrightarrow & G^{(1)} & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & \mu_2 & \longrightarrow & 0
\end{array}$$

— $j_{1*}$ induces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G^{(1)} & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & \mu_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \epsilon_2 & \longrightarrow & H' & \longrightarrow & \mu_2 & \longrightarrow & 0
\end{array}$$

— diagramm chasing gives $2\mathrm{id}_D = 0$ & 6-terms Hom – Ext

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_2 & \longrightarrow & D & \longrightarrow & \mu_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & G^{(1)} & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & \mu_2 & \longrightarrow & 0
\end{array}$$

The upper row is actually the segment of length 2 of $(\mathbb{Q}_2/\mathbb{Z}_2)_{\mathbb{Z}}$

$$0 \longrightarrow \mu_2 \longrightarrow \mu_4 \longrightarrow \mu_2 \longrightarrow 0$$

and we obtained $i_2 = \gamma \circ \delta : \mu_4 \simeq D \longrightarrow G^{(2)}$.

• **There are non-trivial 2-divisible groups of height 2 $/\mathbb{Z}$ but they all are isogeneous to** $(\mathbb{Q}_2/\mathbb{Z}_2)_{\mathbb{Z}} \times (\mathbb{Q}_2/\mathbb{Z}_2)(1)_{\mathbb{Z}}$,

cf. V.Abrashkin, Mat. Zametki (1976)

1.5.1. *The case $g = 1$ of the Shafarevich Conjecture.*

In this case $\mathcal{A}(2)$ is an extension of $(\mathbb{Q}_2/\mathbb{Z}_2)_{\mathbb{Z}}$ via $(\mathbb{Q}_2/\mathbb{Z}_2)(1)_{\mathbb{Z}}$. This extension splits over $\mathbb{F}_2$ (use the existence of a maximal reduced subscheme over perfect fields of positive characteristic) and $\mathcal{A}_{\mathbb{F}_2} = \mathcal{A} \bmod 2$ contains infinitely many $\mathbb{F}_2$-rational points.

Contradiction.

### 1.6. **General approach.**

1.6.1. *2-divisible groups.*

Suppose $\mathcal{G}$ is a 2-div group $/\mathbb{Z}$ and for some $a, b$

$$0 \longrightarrow \mu_{2,\mathbb{Z}}^a \longrightarrow G^{(1)} \longrightarrow (\mathbb{Z}/2)_{\mathbb{Z}}^b \longrightarrow 0 \quad \text{(the main difficulty)}$$

Then $\mathcal{G}$ is an extension of $(\mathbb{Q}_2/\mathbb{Z}_2)_{\mathbb{Z}}^a$ via $(\mathbb{Q}_2/\mathbb{Z}_2)(1)_{\mathbb{Z}}^b$ and there is no abelian scheme $\mathcal{A}$ over $\mathbb{Z}$ such that $\mathcal{A}(2) \simeq \mathcal{G}$ (similarly to the case $g = 1$!)

1.6.2. *The case $g = 2$.*

Here $G = G^{(1)} \in \mathrm{Gr}_{\mathbb{Z}}$, $|G| = 2^4$.

$\mathrm{Gal}(\mathbb{Q}(G)/\mathbb{Q}) \subset \mathrm{GL}_4(\mathbb{F}_2)$ — non-soluble.

$A(G)_{\mathbb{Q}} = \mathbb{Q} \oplus (\underset{\alpha}{\oplus} K_\alpha)$ corresponds to orbits of the $\Gamma_{\mathbb{Q}}$-action on $\mathbb{F}_2^4$.

**Remark.** We can assume that 1.6.1. takes place for ffgs of order 8. If at least one orbit spans a proper subspace in $\mathbb{F}_2^4$ then we get a proper subgroup scheme in $G^{(1)}$ and we obtain property 1.6.1.

The only ways for the lengths of these orbits to consider are:

$1 + 15$, $1 + 5 + 10$, $1 + 5 + 5 + 5$, $1 + 6 + 9$ and $1 + 7 + 8$

(use that the sum of elements in each orbit should be 0, otherwise we shall have a subgroup of order 2)

— all $K_\alpha$ are unramified outside 2;

— $\prod_{\alpha} D(K_\alpha/\mathbb{Q})$ divides $D(A(G)/\mathbb{Z})$

**Tate's formula, [31]:** $v_2(D(A(G))) = n2^h$, $n = \dim G \otimes F_2$

gives $\prod_{\alpha} D(K_\alpha) \leqslant 2^{n2^h} = 2^{32}$;

• Lower bounds come from either the Minkowski or the Odlyzko estimates for $|D(K_\alpha/\mathbb{Q})|^{1/[K:\mathbb{Q}]}$

— Odlyzko's estimates (see for the tables in the appendix):

$[K : \mathbb{Q}] = N \;\Rightarrow\; |D(K)|^{1/N} \geq d_N \to d_\infty$

(Here: $d_\infty \approx 22.352$ and $d_\infty \approx 41.122$ (under GRH).)

— First case: $[K:\mathbb{Q}] = 15$ implies $8.423^{15} = d_{15}^{15} > 2^{32}$;

— Second case: $3.927^5 \cdot 6.585^10 > 2^{32}$;

— Third case: $(d_5^5)^3 = 3.927^{15} < 2^{32}$ — no contradiction:

but such lengths of orbits appear only if all $K_\alpha/\mathbb{Q}$ are cyclic of degree 5 over $\mathbb{Q}$ & unramified outside 2, but such fields do not exist (use class field theory)

— Fourth case: here $d_6^6 d_9^9 = 4.549^6 \cdot 6.134^9 > 2^{32}$

— Fifth case: here $d_7^7 \cdot d_8^8 > 2^{32}$ (can be treated also directly)

## 1.7. **The case $g = 3$.**

Still works with Odlyzko's estimates but requires much more complicated count of orbits, [7]. Resulted in:

- 2-divisible groups $/Z$ of height $\leqslant 6$ are isogeneous to the trivial ones.
- there are no $\mathcal{A}$ over $\mathbb{Z}$ with $g \leqslant 3$.
- Tate's estimate is not good enough for $g \geqslant 4$
- Our hope: index of $A(G)$ in its integral closure should be very big.

### 1.7.1. *What about $p > 2$?*

In this case Tate's estimate is too bad.

But we have even more: if $\mathcal{G}$ is a $p$-divisible group $/\mathbb{Z}$ and

$$0 \longrightarrow \mu_p^a \longrightarrow G^{(1)} \longrightarrow \varepsilon_p^b \longrightarrow 0 \quad \text{(the main difficulty!)}$$

then $G^{(1)} \simeq \mu_p^a \times \varepsilon_p 1^b$ and $\mathcal{G}$ is a trivial $p$-divisible group.

**Remark.** If there is a non-trivial abelian scheme over $\mathbb{Z}$ then we should have examples of non-trivial $p$-div groups over $\mathbb{Z}$ for all primes $p$.

## 1.8. **The Shafarevich Conjecture almost everywhere.**

1.8.1. *AV's with good ordinary reduction at 2.*

— $\mathcal{A}(2)^{(1)} = G \in \mathrm{Gr}_{\mathbb{Z}}$, $G \otimes \bar{\mathbb{F}}_2 = \mu^g_{2,\bar{\mathbb{F}}_2} \times \epsilon^g_{2,\bar{\mathbb{F}}_2}$,

— equiv, over $\bar{O} := W(\bar{\mathbb{F}}_2)$

$$0 \longrightarrow \mu^g_{2,\bar{O}} \longrightarrow G^{(1)} \otimes \bar{O} \longrightarrow \epsilon^g_{2,\bar{O}} \longrightarrow 0$$

— $G \otimes \bar{O} = \sum_{i,j} G_{ij} \in \oplus_{i,j}\mathrm{Ext}(\epsilon_{2,\bar{O}}, \mu_{2,\bar{O}})$,

— Field-of-definition of pts $\mathbb{Q}_{2,ur}(G_{ij}) = \mathbb{Q}_{2,ur}(\sqrt{v_{ij}})$ with $v_{ij} \in \bar{O}^*$

- $\Gamma^{(v)}_{\mathbb{Q}_2}$ acts trivially on $\mathbb{Q}_2(G) \subset \mathbb{Q}_{2,ur}(\{\sqrt{v_{ij}} \mid 1 \leqslant i, j \leqslant g\})$ if $v > 1$;
- $|D(\mathbb{Q}(G))|^{[\mathbb{Q}(G):\mathbb{Q}]^{-1}} < 2^2 < d_4$, $[\mathbb{Q}(G) : \mathbb{Q}] \leqslant 3 \ \Rightarrow \mathbb{Q}(G) \subset \mathbb{Q}(i)$

— at least one of $K_\alpha$ equals $\mathbb{Q}$

So, no such abelian schemes over $\mathbb{Z}$.

1.8.2. *AV's with good ordinary reduction at* $p \geqslant 3$.

— if $\mathcal{A} \otimes \mathbb{F}_p$ is ordinary at $p$, $G = \mathcal{A}(p)^{(1)}$ then

- $\Gamma^{(v)}_{\mathbb{Q}_p}$ acts trivially on $\mathbb{Q}_p(G)$ if $v > 1/(p-1)$;
- $|D(\mathbb{Q}(G)/\mathbb{Q})|^{[\mathbb{Q}(G):\mathbb{Q}]^{-1}} < p^{p/(p-1)}$

For $p \leq 17$:

— $Q(G) \subset \mathbb{Q}(\sqrt[p]{1})$

— e.g. $17^{17/16} = 20.293 < d_{600}$ but $\mathbb{Q}(\sqrt[17]{1}) \subset \mathbb{Q}(G)$, so $16 \cdot 60 > 600$

and we can proceed via class field theory.

**Remark.** $19^{19/18} \approx 22.37 > d_{10^7}$ ???? (under GRH?)

— $G^{(1)}$ is a product of constant etale and mult group schemes $/\mathbb{Z}$;

— $\mathcal{A}(p)$ is trivial and, therefore, no such (non-triv) AV's $\mathcal{A}$ over $\mathbb{Z}$.

**So, the Shafarevich Conjecture holds almost everywhere.**

## 1.9. **The Shafarevich Conjecture, general case.**

Same method but the ramification estimates proved in general situation:

— $G \in \mathrm{Gr}_{W(k)}$, $k$ – perfect of char.$p$, $p\mathrm{id}_G = 0$; Frac $W(k) = K_0$

• $\Gamma_{K_0}^{(1/(p-1))}$ acts trivially on $K_0(G)$;

• $|D(K_0(G)/K_0)|^{1/[K_0(G):K_0]} < p^{1+1/(p-1)}$ (same as for ordinary!)

Two ways to prove the ramification estimate:

a) J.-M.Fontaine: very elegant and general approach:

— $G \in \mathrm{Gr}_{O_K}$, $p^N\mathrm{id}_G = 0$, $[K : K_0] = e$ then

• $\Gamma_K^{(v)}$ acts trivially on $K(G)$ if $v > e(N + e/(p-1)) - 1$

The proof is based on:

— there is an embedding $G \subset$ $p$-div gr$/O_K$;

— rigidity properties of $p$-div groups

b) Alternative way:

— Fontaine's classification of $G \in \mathrm{Gr}_{W(k)}$

identifies the objects of $\mathrm{Gr}_{W(k)}$ with the Fontaine-Laffaille modules with filtration of length 1 (if $p = 2$ we should restrict to the subcategory of unipotent objects)

— improved classification for $p = 2$ (removing the restriction to unipotent objects, cf. [4])

— extract equations for $G \in \mathrm{Gr}_{W(k)}$ s.t. $p\mathrm{id}_G = 0$:

if the FL module of $G$ is given via

$$(\varphi_1(\bar{m}_1), \varphi_0(\bar{m}_0) = (\bar{m}_1, \bar{m}_0)C$$

with $C \in \mathrm{GL}(W(k))$ then

$A(G) = W(k)[\bar{X}_1, \bar{X}_0]$ with equations

• $(-1/p)\bar{X}_1^p, \bar{X}_0^p) = (\bar{X}_1, \bar{X}_0)C$

(seems no good presentation via equations for $p\mathrm{id}_G \neq 0$)

— explicit computations with upper ramification numbers;

**Remark.** In author's overview we mentioned in the very beginning one can find very short new proof of the above ramification estimate; cf. also recent author's paper in the archive.

1.10. **Extras.**

— The above approach could be applied to study the existence of AV's with everywhere good reduction over algebraic number fields with small discriminants, cf. [21]. [5] and, especially,

R.Schoof: There are AV with good reduction everywhere over cyclotomic fields $\mathbb{Q}(\zeta_f)$ if and only if $f \notin \{1,3,4,5,7,8,9,11,12,15\}$. For $f=11,15$ he used estimates under GRH

— A characterization of Galois modules coming from group schemes over $W(k)$, [5]

## 2. Projective varieties over $\mathbb{Q}$ with everywhere good reduction

For projective $Y/\mathbb{C}$, let

$h^N(Y) := \dim_{\mathbb{C}} H^N(Y,\mathbb{C})$, $h^{ij}(Y) = \dim_{\mathbb{C}} H^i(Y, \Omega^j_Y)$.

**Theorem** $X/\mathbb{Q}$ *projective with good reduction everywhere then*

a) $h^1(X_{\mathbb{C}}) = 0$, $h^2(X_{\mathbb{C}}) = h^{11}(X_{\mathbb{C}})$, $h^3(X_{\mathbb{C}}) = 0$;

b) $h^4(X_{\mathbb{C}}) = h^{22}(X_{\mathbb{C}})$ modulo GRH.

2.1. **General approach.**

$X/\mathbb{Q}$ a projective non-singular variety; $p$ — prime, $N \in \mathbb{N}$;

— $V = H^N_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$ — f.dim $/\mathbb{Q}_p$ with continuous $\Gamma_{\mathbb{Q}}$-action.

— $X$ has everywhere good reduction $\Leftrightarrow$

$X = \mathcal{X} \otimes \mathbb{Q}$, $\mathcal{X}$ proper smooth $/\mathbb{Z}$

**Want**: $V = V_0 \supset V_1 \supset \cdots \supset V_{N-1} \supset V_N = 0$

with $\mathbb{Q}_p[\Gamma_{\mathbb{Q}}]$-modules $V_i$, $V_i/V_{i+1} = \mathbb{Q}_p(i)^{s_i}$, $s_i \geqslant 0$

Then

— $\forall l \neq p$, $\mathrm{Frob}_l$ of $\mathcal{X} \otimes \mathbb{F}_l$ acts on $V_i/V_{i+1}$ via mult by $l^i$

— Riemann Conjecture over $\mathbb{F}_l$ (proved by Deligne) $\Rightarrow$ all $l^i = l^{N/2}$;

(use that $H^i_{et}(X, \mathbb{Q}_p) = H^i_{et}(\mathcal{X} \otimes \mathbb{F}_l, \mathbb{Q}_p))$

— Comparison to the De Rham cohomology:

if $N$ is odd $\Rightarrow h^N = \dim V = 0$.

if $N = 2N_1$ is even $\Rightarrow V \simeq \mathbb{Q}_p(N_1)^s$, $h^N = h^{N_1,N_1}$.

*Achieving* **"Want"**:

**Step** *I*. *X has good reduction modulo* $l \neq p \Rightarrow$ *V unramified at* $l$.

(use again $H^i_{et}(X, \mathbb{Q}_p) = H^i_{et}(\mathcal{X} \otimes \mathbb{F}_l, \mathbb{Q}_p)$)

$T$ — $\Gamma_{\mathbb{Q}}$-inv lattice in $V$, $\mathbb{Q}_T$ is the f-of-def of points $T/pT$ then

• $D(\mathbb{Q}_T/\mathbb{Q})$ is a **power of** $p$.

**Step** $II_{crys}$. *X good reduction* $\mathrm{mod}\, p \Rightarrow V$ *crystalline* $\Gamma_{\mathbb{Q}_p}$*-module*

(via some embedding $\Gamma_{\mathbb{Q}_p} \subset \Gamma_{\mathbb{Q}}$)

• $N \leqslant p-1 \Rightarrow V$ comes from Fontaine-Laffaille theory, [19].

• $N \leqslant p-2 \Rightarrow$ all finite subquotients of $V$ come from FL theory (here the FL functor is fully faithful);

(if $N = p-1 \Rightarrow$ we need the modification of the FL theory from [6])

• $\mathbb{Q}(p, N) \supset \mathbb{Q}_T$ is the f-of-def of points of all $T/pT \Rightarrow$

$\Gamma^{(v)}_{\mathbb{Q}_p}$ acts trivially on $\mathbb{Q}_p(p, N)$ if $v > N/(p-1)$

(ramification estimates)

• $|D(\mathbb{Q}(p, N)/\mathbb{Q})|^{[K:\mathbb{Q}]^{-1}} < p^{1+N/(p-1)}$ (ready for Odlyzko)

**Step** $III_{crys}$. *Odlyzko's estimates*:
Reminder: $[K : \mathbb{Q}] = N \Rightarrow |D(K/\mathbb{Q})|^{1/N} \geq d_N \underset{N\to\infty}{\to} d_\infty$

— $d_\infty \approx 22.352$ and $d_\infty \approx 41.122$ (under GRH).

— $p = 5$, $N = 1, 2, 3$, then $5^{1+3/4} = 16.7185... < d_{240} = 18.788$

• $\mathrm{Gal}(\mathbb{Q}(5, 3)/\mathbb{Q})$ is soluble, $\mathbb{Q}(5, 3) \subset \mathbb{Q}(\zeta_5, \sqrt[5]{\zeta_5 + \zeta_5^{-1}})$.

• $N = 4$ then (under GRH) $\mathbb{Q}(5, 4)$ is still totally ramified at 5 with the same maximal tamely ramified subfield $\mathbb{Q}_5(\zeta_5)$.

**Step** $IV_{crys}$. $\Gamma_{\mathbb{Q}}$*-equivariant filtration for* $T/p$

Suppose for some $(p, N)$ where $N \leqslant p-1$ (our case is $p = 5, N \leq 4$):

$\mathbb{Q}(p, N)$ is totaly ramified at $p \Rightarrow$

• {global behaviour of $T/p$} $\equiv$ {local behaviour of $T/p$ over $\mathbb{Q}_{p,ur}$}

— over $p$, $\mathbb{Q}(p, N)_{tr} = \mathbb{Q}(\zeta_p) \Rightarrow$

• $\Gamma_{\mathbb{Q}}$-filtration $T/p = H_0 \supset H_1 \supset \cdots \supset H_N = 0$
with $H_i/H_{i+1} \simeq \mathbb{F}_p(i)^{s_i}$

• (Crucial!) $\mathrm{Ext}_{glob}(\mathbb{F}_p(i), \mathbb{F}_p(j)) = \mathrm{Ext}_{p\text{-crys}}(\mathbb{F}_p(i), F_p(j)) = 0$ if $i \geqslant j$

— if $N \leq p-2$ use Fontaine-Laffaille theory

— if $N = p - 1$ use modification of FL theory

this modification takes values in the category of filtered Galois modules, e.g. there are two simple objects

$\mathbb{F}_p(0) = (\mathbb{F}_p, \mathbb{F}_p)$ and $\mathbb{F}_p(p-1) = (\mathbb{F}_p, 0)$).

**Step** $V_{crys}$.

$\Gamma_{\mathbb{Q}}$-*filtration* $V = V_0 \supset V_1 \supset \cdots \supset V_N = 0$ *with* $V_i/V_{i+1} = \mathbb{Q}_p(i)^{s_i}$

— devissage in pre-abelian categories, cf. [7,11]

## 3. Semi-stable case of the Shafarevich Conjecture

**Theorem** $X/\mathbb{Q}$ *projective non-singular with good reduction at* $l \neq 3$ *and semi-stable reduction at 3 then* $h^2(X_{\mathbb{C}}) = h^{11}(X_{\mathbb{C}})$.

### 3.1. General approach.

— AV's over $\mathbb{Q}$ with bad semi-st reduction modulo one small prime:

Brumer-Kramer, Schoof, Verhoek etc ("tr" ramification at bad reduction + cryst in others)

Again choose a prime $p$ and $N \in \mathbb{N}$;

— $V = H^N_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$ — f.dim $/\mathbb{Q}_p$ with continuous $\Gamma_{\mathbb{Q}}$-action.

Want:

— if $N$ is odd $\Rightarrow h^N = \dim V = 0$.

— if $N = 2N_1$ is even $\Rightarrow V \simeq \mathbb{Q}_p(N_1)^s$, $h^N = h^{N_1,N_1}$.

**Steps** $II_{sst}, III_{sst}, IV_{sst}$.

$X$ has s-st reduction at $p \Rightarrow V$ s-st $\Gamma_{\mathbb{Q}_p}$-module (via some $\Gamma_{\mathbb{Q}_p} \subset \Gamma_{\mathbb{Q}}$)

$T$ — $\Gamma_{\mathbb{Q}}$-inv lattice in $V$,

$\mathbb{Q}_{st}(p, N)$ — the field-of-definition of pts of $T/p$.

- $N < p \Rightarrow V$ can be described via Breuil's theory.
- $N \leqslant p - 2 \Rightarrow$ Breuil's theory describes $T/p$ over $\mathbb{Q}_p$;
- $N = p - 1 \Rightarrow$ modification of Breuil's theory describes $T/p$ over $\mathbb{Q}_{p,ur}$.
- $|D(\mathbb{Q}_{st}(p, N)/\mathbb{Q}))|^{[\mathbb{Q}_{st}(p,N):\mathbb{Q}]^{-1}} < p^{2+N/(p-1)-1/p}$.

— $N = 2$, $p = 5 \Rightarrow 5^{2+1/2-1/5} > d_\infty$ (even under GRH)

- can't apply Breuil's theory

• $N = 2$, $p = 3$ (use modification!), $3^{3-1/3} < d_{238}$ —soluble case!

$[\mathbb{Q}_{st}(3,2) : \mathbb{Q}] < d_{238}$, $\mathbb{Q}_{st}(3,2) = \mathbb{Q}(\zeta_9, \sqrt[3]{3})$.

**Computational Part — SAGE**

**Theorem** $\mathbb{Q}_{st}(3,2) = K_1 = \mathbb{Q}(\zeta_9, \sqrt[3]{3})$

a) $h_{K_1} = 1$;

b) $K_1(\sqrt[3]{\varepsilon}) :$ $\{8$ fund. units & $\zeta_9\}$; let $\pi \in K_1$, $(\pi^{18}) = (3)$.

Non-trivial extension $K_1(\sqrt[3]{\eta})$ inside $K(3,2)$ means

(quiet typical for all Schoof's papers)

$\eta \equiv 1 \bmod (\pi^{19})$ (use ramification estimate)

BUT there is a basis $\varepsilon_i$, $1 \leqslant 9$, of units modulo cubes such that

$18 v_3(\varepsilon_i - 1) = 1, 2, 4, 5, 7, 8, 11, 13, 16.$

**Step** $V_{st}$.

Similar to $V_{crys}$ but $\mathrm{Ext}_{3\text{-sst}}(\mathbb{F}_3(1), \mathbb{F}_3(1)) = \mathbb{F}_3 \neq 0$.

We overcome this problem by proving that such subquotient never appears as a subquotient of any such 3-divisible group.

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Appendix. Odlyzko tables for lower bounds of the root discriminants

            of algebraic number fields



                 Table 1: GRH BOUNDS FOR DISCRIMINANTS
```

| n | b | D^{1/n} | b | D^{1/n} |
|---|---|---|---|---|
| 1 | 0.340 | 0.996 | 0.300 | 0.874 |
| 2 | 0.700 | 2.225 | 0.580 | 1.721 |
| 3 | 1.050 | 3.630 | 0.800 | 2.519 |
| 4 | 1.350 | 5.124 | 1.050 | 3.263 |
| 5 | 1.550 | 6.640 | 1.200 | 3.954 |
| 6 | 1.750 | 8.143 | 1.350 | 4.592 |
| 7 | 1.900 | 9.611 | 1.500 | 5.185 |
| 8 | 2.050 | 11.036 | 1.600 | 5.734 |
| 9 | 2.200 | 12.410 | 1.700 | 6.247 |
| 10 | 2.300 | 13.736 | 1.800 | 6.726 |
| 11 | 2.400 | 15.012 | 1.900 | 7.176 |
| 12 | 2.500 | 16.238 | 2.000 | 7.598 |
| 13 | 2.550 | 17.422 | 2.050 | 7.997 |

| | | | | |
|---|---|---|---|---|
| 14 | 2.650 | 18.559 | 2.100 | 8.371 |
| 15 | 2.700 | 19.657 | 2.200 | 8.730 |
| 16 | 2.800 | 20.711 | 2.250 | 9.068 |
| 17 | 2.850 | 21.734 | 2.300 | 9.390 |
| 18 | 2.900 | 22.720 | 2.400 | 9.697 |
| 19 | 2.950 | 23.672 | 2.450 | 9.990 |
| 20 | 3.000 | 24.594 | 2.500 | 10.270 |
| 21 | 3.100 | 25.474 | 2.550 | 10.539 |
| 22 | 3.100 | 26.351 | 2.550 | 10.797 |
| 23 | 3.100 | 27.178 | 2.600 | 11.045 |
| 24 | 3.200 | 28.001 | 2.650 | 11.283 |
| 25 | 3.200 | 28.787 | 2.700 | 11.512 |
| 26 | 3.300 | 29.554 | 2.750 | 11.733 |
| 28 | 3.300 | 31.020 | 2.800 | 12.153 |
| 30 | 3.400 | 32.425 | 2.850 | 12.545 |
| 32 | 3.500 | 33.750 | 2.950 | 12.912 |
| 34 | 3.500 | 35.005 | 3.000 | 13.258 |
| 36 | 3.600 | 36.219 | 3.000 | 13.581 |
| 38 | 3.700 | 37.356 | 3.100 | 13.894 |
| 40 | 3.700 | 38.471 | 3.100 | 14.183 |
| 42 | 3.800 | 39.514 | 3.200 | 14.465 |
| 44 | 3.800 | 40.542 | 3.200 | 14.728 |
| 46 | 3.800 | 41.504 | 3.300 | 14.984 |
| 48 | 3.900 | 42.456 | 3.300 | 15.225 |
| 50 | 3.900 | 43.356 | 3.400 | 15.456 |
| 52 | 4.000 | 44.230 | 3.400 | 15.680 |
| 56 | 4.000 | 45.884 | 3.500 | 16.097 |
| 60 | 4.100 | 47.452 | 3.500 | 16.482 |
| 64 | 4.200 | 48.913 | 3.600 | 16.846 |
| 68 | 4.200 | 50.285 | 3.700 | 17.180 |
| 72 | 4.300 | 51.601 | 3.700 | 17.497 |
| 76 | 4.400 | 52.822 | 3.800 | 17.793 |
| 80 | 4.400 | 54.014 | 3.800 | 18.073 |
| 84 | 4.500 | 55.119 | 3.900 | 18.338 |
| 88 | 4.500 | 56.204 | 3.900 | 18.589 |
| 92 | 4.500 | 57.214 | 4.000 | 18.826 |
| 96 | 4.600 | 58.205 | 4.000 | 19.055 |
| 100 | 4.600 | 59.141 | 4.000 | 19.268 |
| 110 | 4.700 | 61.335 | 4.100 | 19.770 |
| 120 | 4.800 | 63.335 | 4.200 | 20.221 |
| 130 | 4.900 | 65.169 | 4.300 | 20.631 |
| 140 | 5.000 | 66.853 | 4.400 | 21.003 |
| 150 | 5.000 | 68.426 | 4.400 | 21.345 |
| 160 | 5.100 | 69.897 | 4.500 | 21.666 |
| 170 | 5.200 | 71.255 | 4.600 | 21.959 |

| | | | | |
|---|---|---|---|---|
| 180 | 5.200 | 72.553 | 4.600 | 22.236 |
| 190 | 5.300 | 73.760 | 4.700 | 22.493 |
| 200 | 5.300 | 74.909 | 4.700 | 22.735 |
| 220 | 5.400 | 77.026 | 4.800 | 23.178 |
| 240 | 5.500 | 78.943 | 4.900 | 23.575 |
| 260 | 5.600 | 80.689 | 5.000 | 23.934 |
| 280 | 5.700 | 82.283 | 5.100 | 24.258 |
| 300 | 5.700 | 83.775 | 5.100 | 24.560 |
| 320 | 5.800 | 85.155 | 5.200 | 24.838 |
| 340 | 5.900 | 86.424 | 5.200 | 25.091 |
| 360 | 5.900 | 87.642 | 5.300 | 25.332 |
| 380 | 6.000 | 88.760 | 5.400 | 25.552 |
| 400 | 6.000 | 89.833 | 5.400 | 25.763 |
| 480 | 6.200 | 93.555 | 5.600 | 26.485 |
| 600 | 6.400 | 97.979 | 5.800 | 27.328 |
| 720 | 6.600 | 101.488 | 6.000 | 27.984 |
| 840 | 6.800 | 104.361 | 6.100 | 28.515 |
| 960 | 6.900 | 106.815 | 6.300 | 28.961 |
| 1000 | 6.900 | 107.548 | 6.300 | 29.094 |
| 1200 | 7.200 | 110.728 | 6.500 | 29.673 |
| 1332 | 7.200 | 112.575 | 6.600 | 29.992 |
| 2400 | 7.800 | 122.112 | 7.200 | 31.645 |
| 4800 | 8.400 | 132.020 | 7.800 | 33.298 |
| 4840 | 8.600 | 132.126 | 7.800 | 33.315 |
| 8862 | 9.200 | 139.766 | 8.400 | 34.541 |
| 10000 | 9.200 | 141.218 | 8.600 | 34.768 |
| 31970 | 10.400 | 153.252 | 9.800 | 36.613 |
| 100000 | 11.600 | 162.651 | 10.800 | 37.994 |
| 254228 | 12.500 | 168.971 | 11.800 | 38.895 |

Table 2: UNCONDITIONAL BOUNDS FOR DISCRIMINANTS

| n | b | D^{1/n} | b | D^{1/n} |
|---|---|---|---|---|
| 1 | 0.420 | 0.996 | 0.360 | 0.874 |
| 2 | 0.900 | 2.222 | 0.700 | 1.719 |
| 3 | 1.350 | 3.609 | 1.000 | 2.513 |
| 4 | 1.750 | 5.062 | 1.300 | 3.250 |
| 5 | 2.050 | 6.514 | 1.550 | 3.927 |

| | | | | |
|---|---|---|---|---|
| 6 | 2.350 | 7.926 | 1.750 | 4.549 |
| 7 | 2.600 | 9.279 | 1.950 | 5.121 |
| 8 | 2.850 | 10.568 | 2.100 | 5.646 |
| 9 | 3.100 | 11.787 | 2.300 | 6.134 |
| 10 | 3.300 | 12.941 | 2.450 | 6.585 |
| 11 | 3.400 | 14.034 | 2.600 | 7.004 |
| 12 | 3.600 | 15.068 | 2.750 | 7.395 |
| 13 | 3.800 | 16.044 | 2.900 | 7.760 |
| 14 | 3.900 | 16.971 | 3.000 | 8.102 |
| 15 | 4.000 | 17.849 | 3.100 | 8.423 |
| 16 | 4.200 | 18.684 | 3.200 | 8.725 |
| 17 | 4.300 | 19.479 | 3.300 | 9.010 |
| 18 | 4.400 | 20.234 | 3.400 | 9.280 |
| 19 | 4.600 | 20.954 | 3.500 | 9.536 |
| 20 | 4.700 | 21.642 | 3.600 | 9.779 |
| 21 | 4.800 | 22.299 | 3.700 | 10.010 |
| 22 | 4.900 | 22.929 | 3.800 | 10.229 |
| 23 | 5.000 | 23.531 | 3.900 | 10.438 |
| 24 | 5.100 | 24.109 | 4.000 | 10.638 |
| 25 | 5.200 | 24.664 | 4.000 | 10.829 |
| 26 | 5.300 | 25.196 | 4.100 | 11.013 |
| 28 | 5.400 | 26.203 | 4.300 | 11.357 |
| 30 | 5.600 | 27.138 | 4.400 | 11.675 |
| 32 | 5.800 | 28.008 | 4.500 | 11.969 |
| 34 | 5.900 | 28.821 | 4.700 | 12.243 |
| 36 | 6.100 | 29.582 | 4.800 | 12.498 |
| 38 | 6.200 | 30.298 | 4.900 | 12.737 |
| 40 | 6.300 | 30.971 | 5.000 | 12.961 |
| 42 | 6.500 | 31.607 | 5.100 | 13.172 |
| 44 | 6.600 | 32.209 | 5.200 | 13.371 |
| 46 | 6.700 | 32.778 | 5.300 | 13.558 |
| 48 | 6.800 | 33.319 | 5.400 | 13.736 |
| 50 | 7.000 | 33.832 | 5.600 | 13.905 |
| 52 | 7.000 | 34.322 | 5.600 | 14.066 |
| 56 | 7.200 | 35.233 | 5.800 | 14.364 |
| 60 | 7.400 | 36.067 | 6.000 | 14.634 |
| 64 | 7.600 | 36.834 | 6.100 | 14.883 |
| 68 | 7.800 | 37.541 | 6.300 | 15.112 |
| 72 | 8.000 | 38.197 | 6.400 | 15.323 |
| 76 | 8.200 | 38.806 | 6.600 | 15.520 |
| 80 | 8.400 | 39.373 | 6.700 | 15.702 |
| 84 | 8.600 | 39.903 | 6.900 | 15.872 |
| 88 | 8.600 | 40.402 | 7.000 | 16.032 |
| 92 | 8.800 | 40.871 | 7.000 | 16.180 |
| 96 | 9.000 | 41.312 | 7.200 | 16.321 |

| | | | | |
|---|---|---|---|---|
| 100 | 9.200 | 41.728 | 7.400 | 16.454 |
| 110 | 9.400 | 42.678 | 7.600 | 16.756 |
| 120 | 9.800 | 43.513 | 7.800 | 17.020 |
| 130 | 10.000 | 44.256 | 8.200 | 17.255 |
| 140 | 10.400 | 44.921 | 8.400 | 17.466 |
| 150 | 10.600 | 45.522 | 8.600 | 17.655 |
| 160 | 10.800 | 46.067 | 8.800 | 17.826 |
| 170 | 11.200 | 46.565 | 9.000 | 17.982 |
| 180 | 11.400 | 47.021 | 9.200 | 18.125 |
| 190 | 11.600 | 47.444 | 9.400 | 18.257 |
| 200 | 11.800 | 47.833 | 9.600 | 18.379 |
| 220 | 12.000 | 48.530 | 10.000 | 18.597 |
| 240 | 12.500 | 49.142 | 10.200 | 18.788 |
| 260 | 13.000 | 49.680 | 10.600 | 18.955 |
| 280 | 13.500 | 50.156 | 10.800 | 19.104 |
| 300 | 13.500 | 50.588 | 11.200 | 19.237 |
| 320 | 14.000 | 50.977 | 11.400 | 19.358 |
| 340 | 14.000 | 51.328 | 11.600 | 19.467 |
| 360 | 14.500 | 51.652 | 11.800 | 19.567 |
| 380 | 15.000 | 51.947 | 12.000 | 19.658 |
| 400 | 15.000 | 52.221 | 12.500 | 19.742 |
| 480 | 16.000 | 53.130 | 13.000 | 20.023 |
| 600 | 17.000 | 54.122 | 14.000 | 20.329 |
| 720 | 18.000 | 54.842 | 15.000 | 20.551 |
| 840 | 19.000 | 55.396 | 16.000 | 20.722 |
| 960 | 20.000 | 55.837 | 17.000 | 20.856 |
| 1000 | 21.000 | 55.966 | 17.000 | 20.895 |
| 1200 | 22.000 | 56.500 | 18.000 | 21.059 |
| 1332 | 23.000 | 56.780 | 19.000 | 21.144 |
| 2400 | 28.000 | 58.061 | 23.000 | 21.535 |
| 4800 | 35.000 | 59.069 | 29.000 | 21.843 |
| 4840 | 35.000 | 59.079 | 29.000 | 21.845 |
| 8862 | 42.500 | 59.655 | 35.000 | 22.021 |
| 10000 | 45.000 | 59.746 | 37.500 | 22.049 |
| 31970 | 67.500 | 60.332 | 57.500 | 22.226 |
| 100000 | 75.000 | 60.582 | 75.000 | 22.308 |
| 254228 | 75.000 | 60.656 | 75.000 | 22.335 |
| 1000000 | 75.000 | 60.691 | 75.000 | 22.348 |
| 2391978 | 75.000 | 60.698 | 75.000 | 22.350 |
| 10000000 | 75.000 | 60.702 | 75.000 | 22.352 |

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