Computational Approach to Riemann Surfaces

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Theta-function solutions to the Kadomtsev-Petviashvili equation

- E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev. Algebro- geometric approach to nonlinear integrable problems. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1994.
- $$3u_{yy} + \partial_x(6uu_x + u_{xxx} - 4u_t) = 0$$
  weakly two-dimensional waves in shallow water
- almost periodic solutions in terms of theta functions on arbitrary compact Riemann surfaces (Krichever 1978)
  $$u = 2\partial_x^2 \ln \Theta(\mathbf{U}x + \mathbf{V}y + \mathbf{W}t + \mathbf{D}) + 2c$$

- $\mathbf{D} \in \mathbb{R}^g$ arbitrary

- Riemann theta function

$$\Theta(\mathbf{x}|\mathbf{B}) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left\{\mathrm{i}\pi\langle\mathbf{Bn},\mathbf{n}\rangle + 2\pi\mathrm{i}\langle\mathbf{n},\mathbf{x}\rangle\right\}$$

- $\mathbf{B}$ Riemann matrix, matrix of b-periods of the holomorphic differentials

- $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, vectors expressible in terms of derivatives of the holomorphic differentials, c constant expressible in terms of theta functions

Hyperelliptic solutions (g=2)

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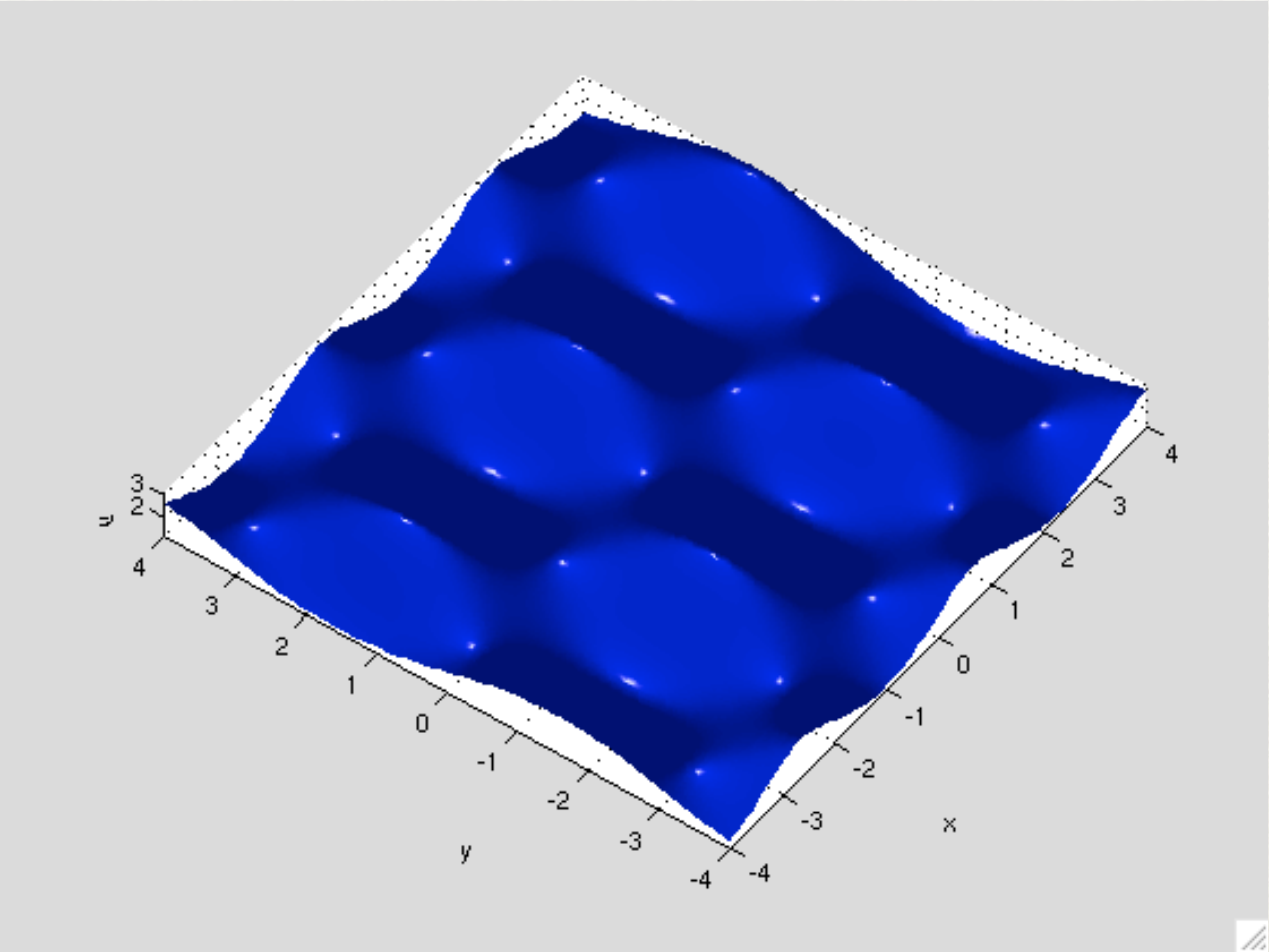

Hyperelliptic solution (g=4)

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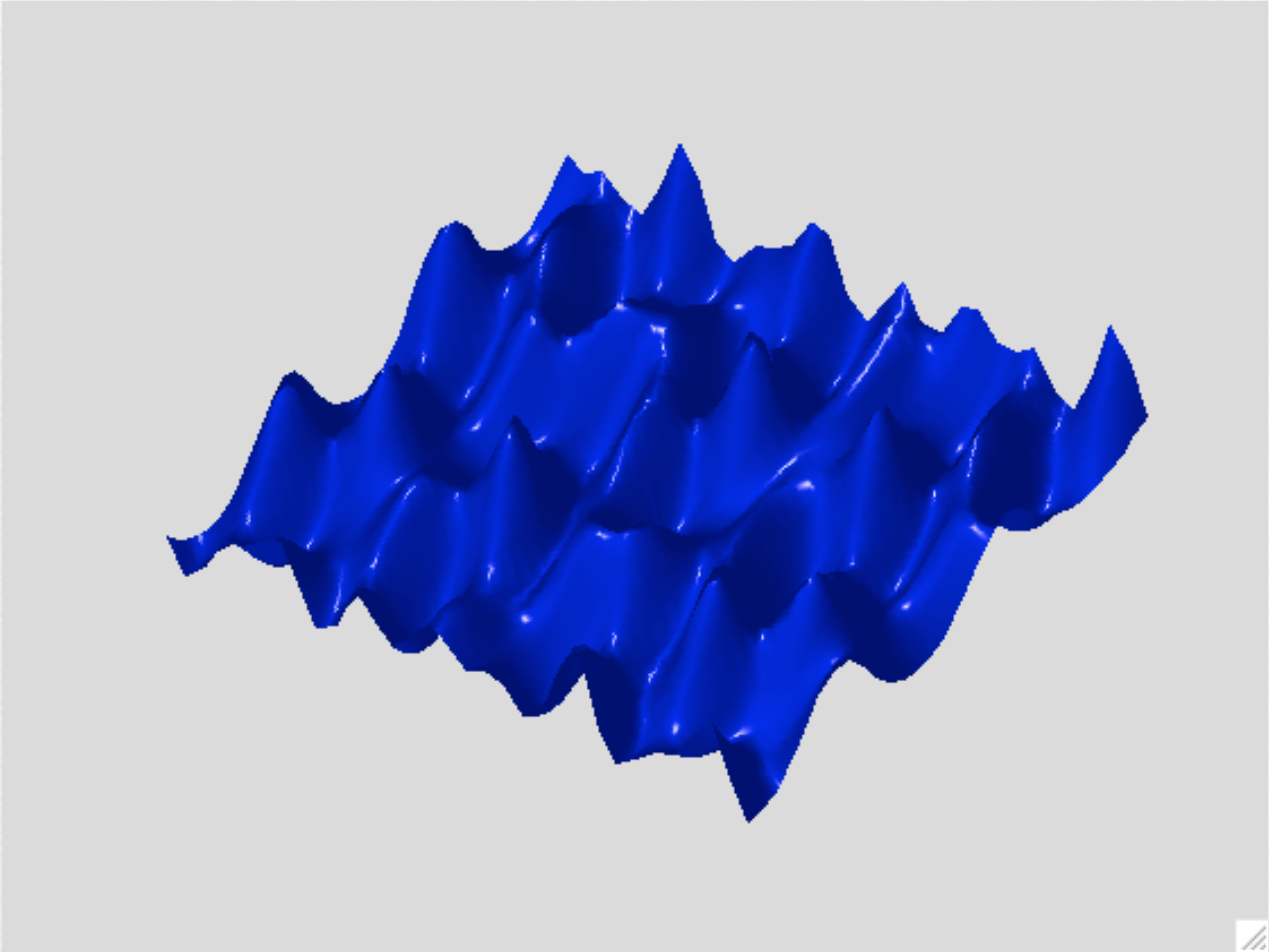

(Line)Solitons (localized in one direction), 2-soliton

- branch points coincide pairwise, surface of genus 0 in the limit

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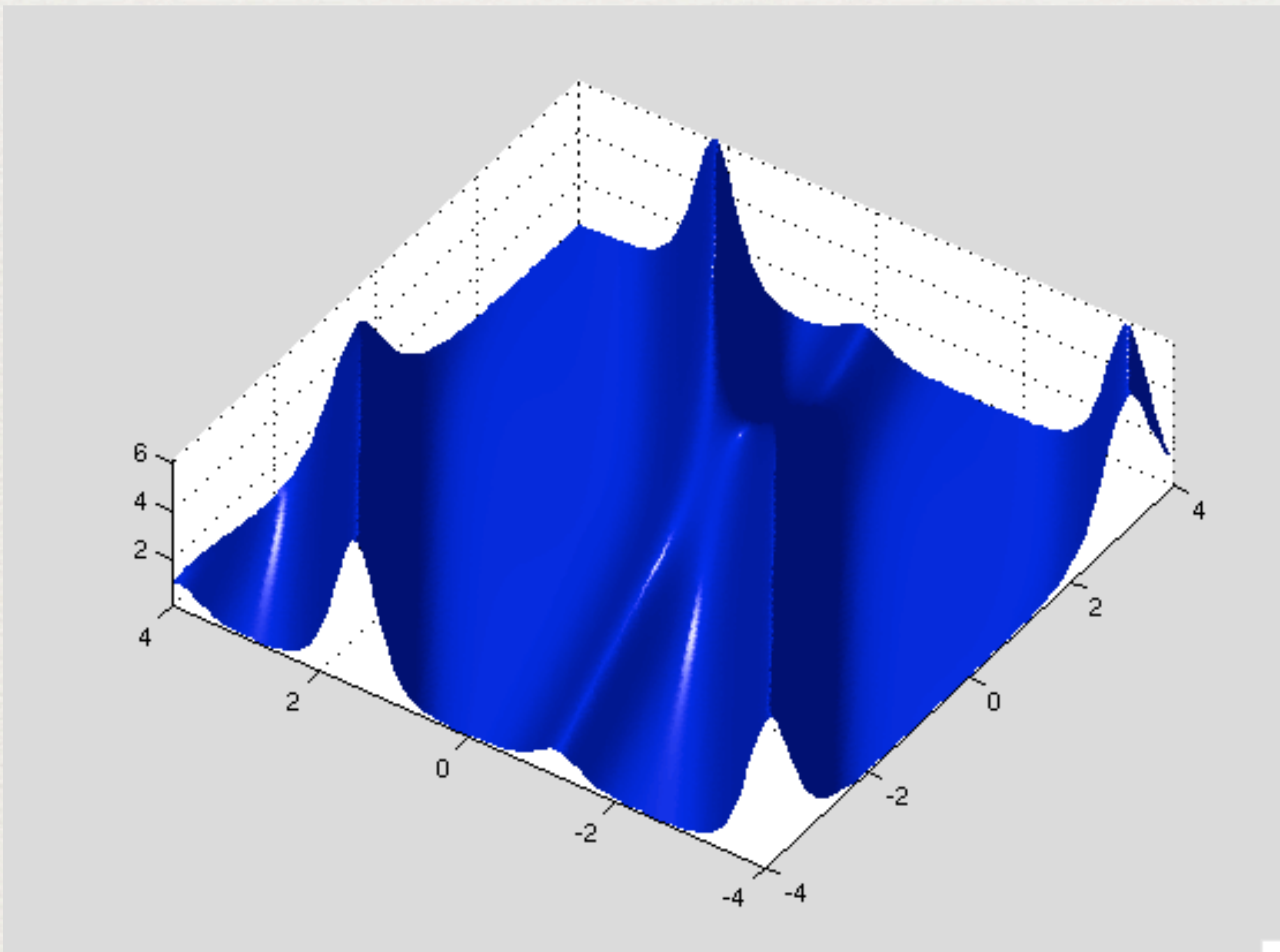

(Line)Solitons, 4-soliton

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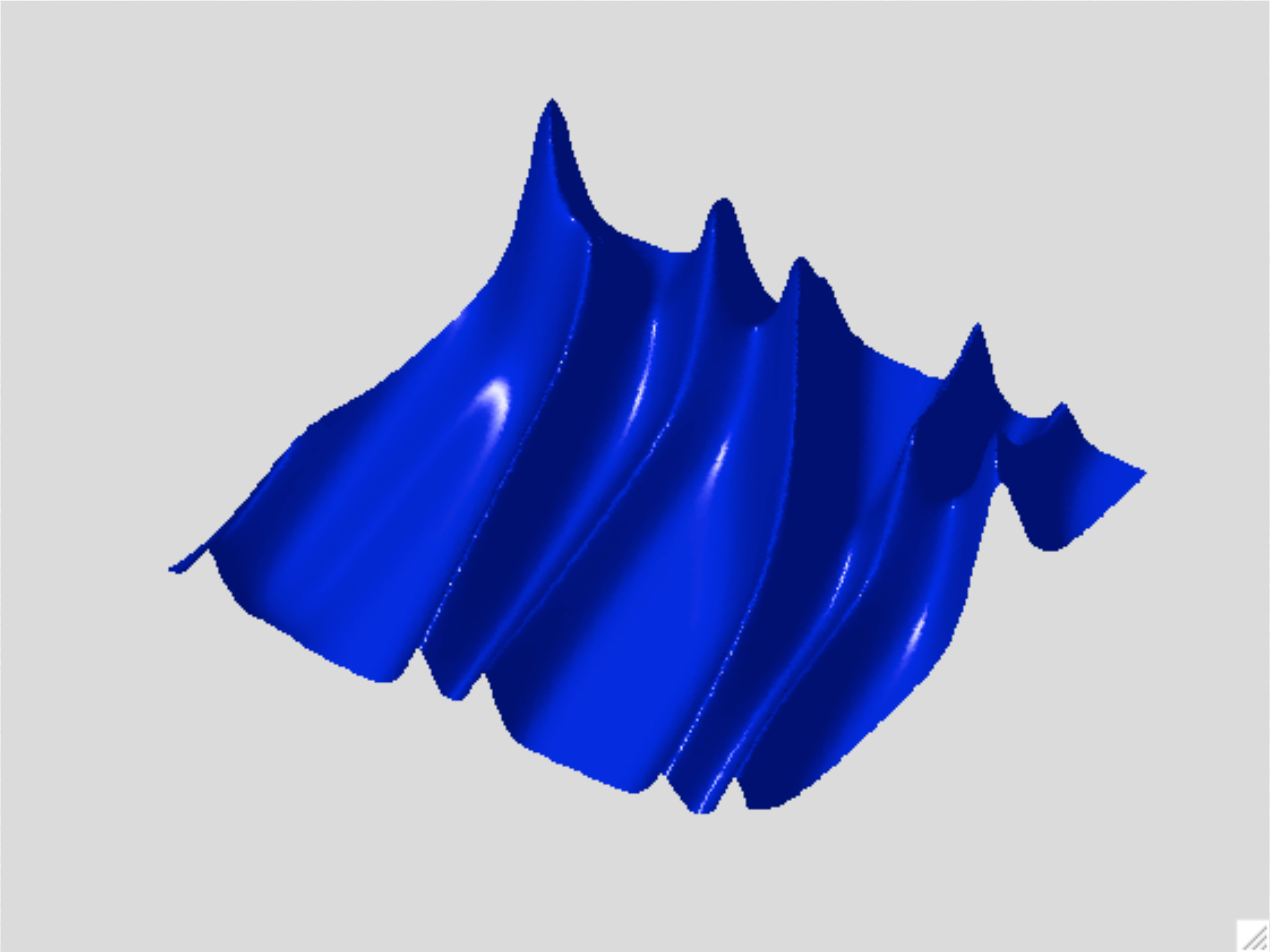


Symbolic vs. numerical

- Bobenko, Bordag (1989), Schottky uniformizations
- Deconinck, v. Hoeij, Patterson: *algcurves* package in Maple (2001), symbolic approach, exact expressions (e.g. RootOf(x^2-2)) manipulated and numerically evaluated, in principle infinite precision
- Frauendiener, K.: fully numeric approach (*floating point)*, hyperelliptic curves (1998), much more rapid, allows study of families of curves and of more complicated curves

Ernst equation and Bianchi surfaces

- Bianchi: Gauss-Weingarten equations with spectral parameter
- Ernst equation: Maison, Belinski-Zakharov 1978

$$\Phi_{,\xi} = \frac{\mathcal{J}_{,\xi}\mathcal{J}^{-1}}{1-\gamma}\Phi\,, \quad \Phi_{,\bar{\xi}} = \frac{\mathcal{J}_{,\bar{\xi}}\mathcal{J}^{-1}}{1+\gamma}\Phi$$

where $\xi = \zeta - i\rho$ and

$$\mathcal{J} = \frac{1}{f}\begin{pmatrix} 1 & -b \\ -b & f^2 + b^2 \end{pmatrix}$$

$$\gamma = \frac{2}{\xi - \bar{\xi}}\left(K - \frac{\xi + \bar{\xi}}{2} + \sqrt{(K - \xi)(K - \bar{\xi})}\right)$$

- spectral parameter K on family of Riemann surfaces of genus 0, branch points dependent on physical coordinates (non-autonomous system), Geroch group

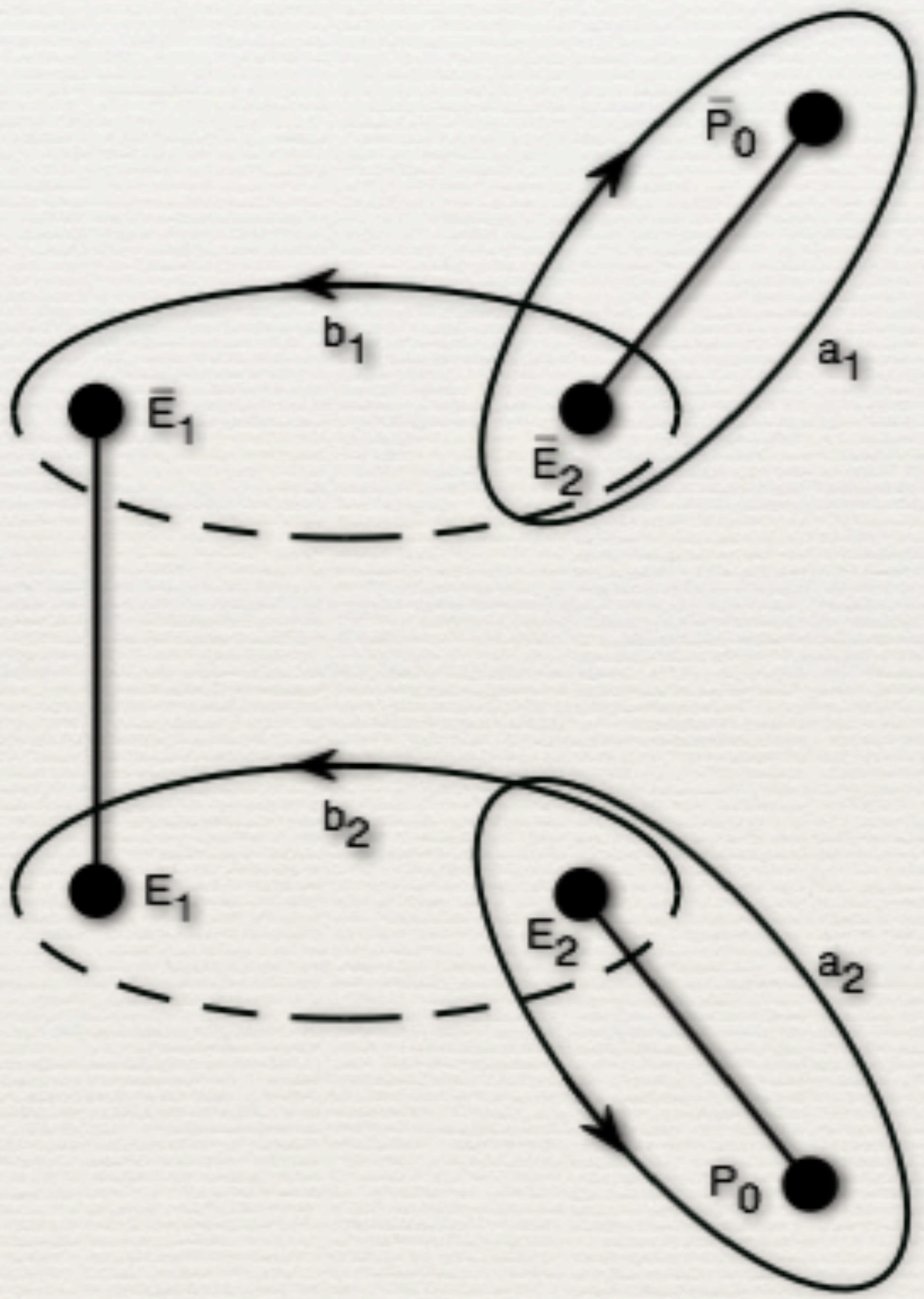
$$\mu^2 = (K - \xi)(K - \bar{\xi})$$

- 2-soliton solution: Kerr black hole

$$\mathcal{E} = \frac{\mathrm{e}^{-\mathrm{i}\varphi} r_+ + \mathrm{e}^{\mathrm{i}\varphi} r_- - 2m\cos\varphi}{\mathrm{e}^{-\mathrm{i}\varphi} r_+ + \mathrm{e}^{\mathrm{i}\varphi} r_- + 2m\cos\varphi}$$

where $r_\pm = \sqrt{(\zeta \pm m\cos\varphi)^2 + \rho^2}$; mass m, angular momentum $J = m^2 \sin\varphi$, horizon on the axis $[-m\cos\varphi, m\cos\varphi]$, $\varphi = 0$: static, spherically symmetric Schwarzschild solution, $\varphi = \pi/2$: extreme Kerr solution (degenerate horizon)

Canonical Cycles (g=2)



$$P_0 = \zeta - i\rho$$

Ernst potential, Newtonian-ultrarelativistic

Korotkin 1989, K., Richter 1998

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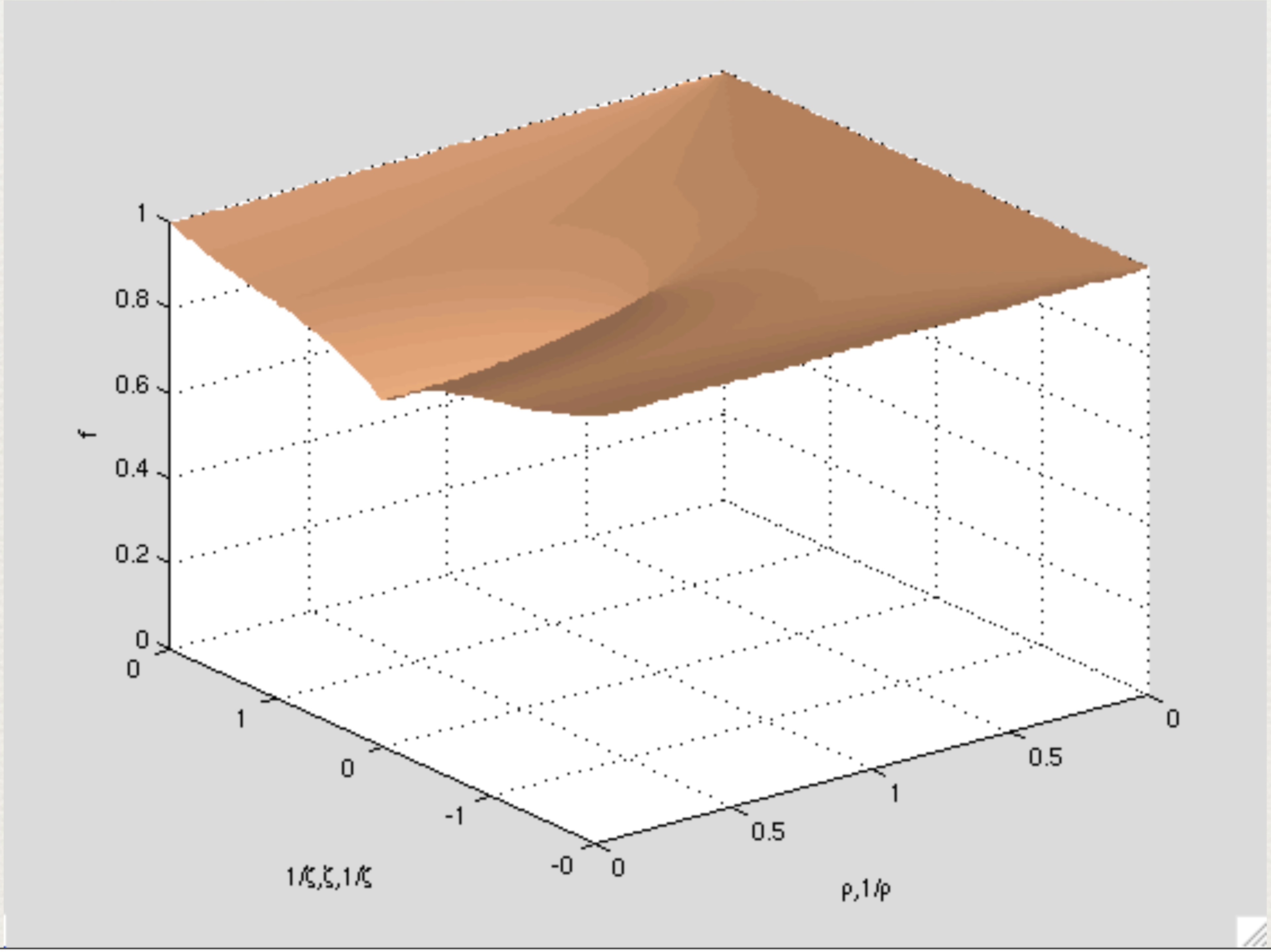

Ernst potential, imaginary part

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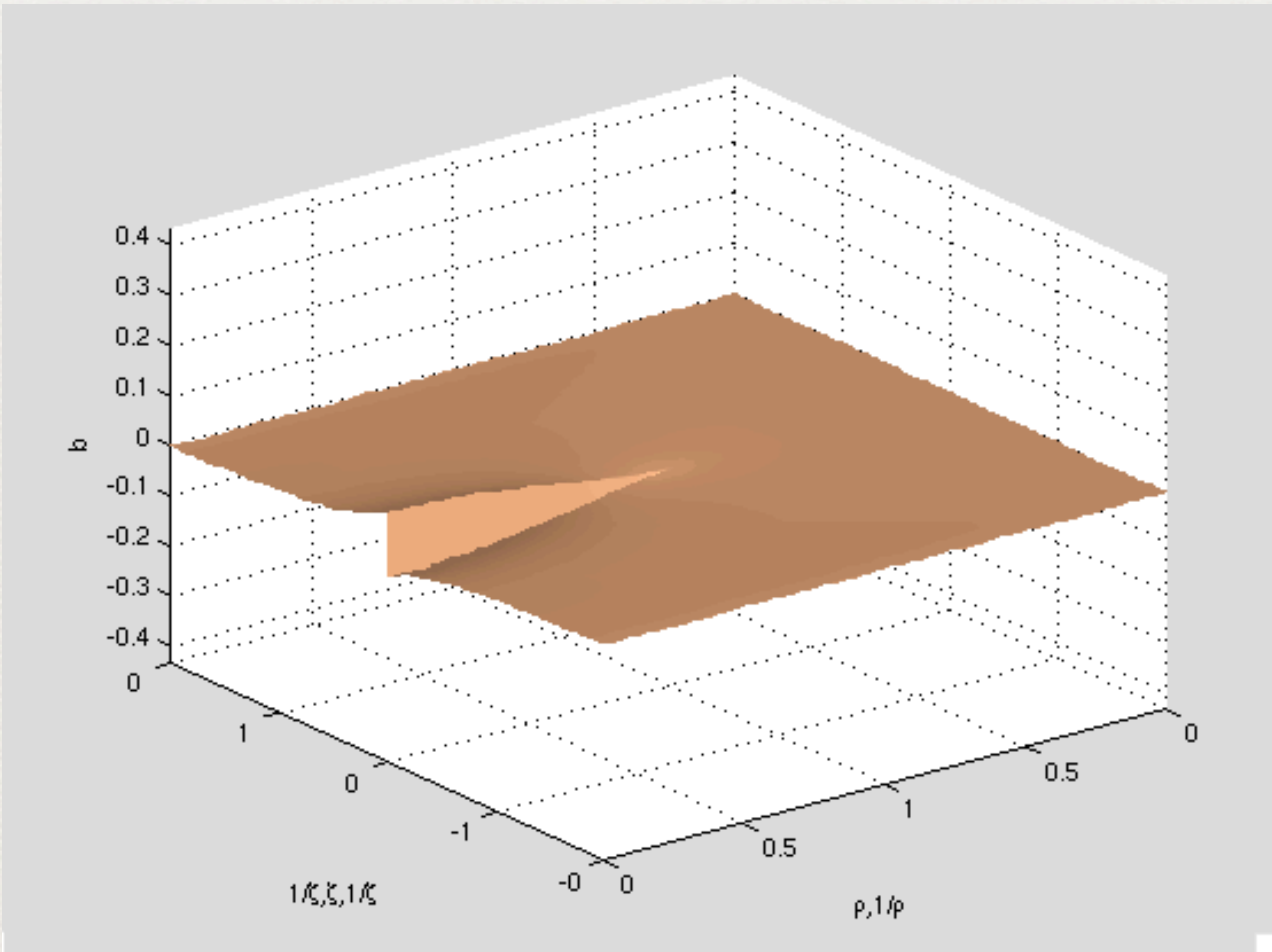


Outline

- Riemann surfaces and algebraic curves
- Branch points and singular points
- Monodromy and homology
- Puiseux expansions and holomorphic differentials
- Real Riemann surfaces
- Theta functions
- Performance tests and examples

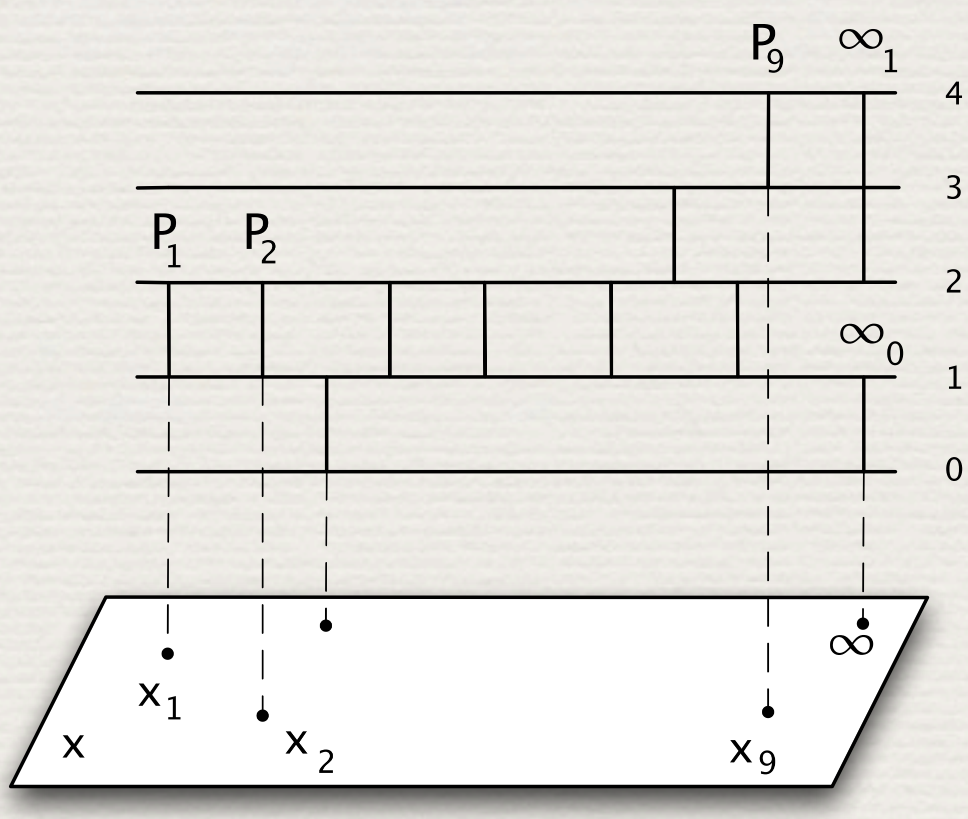
Riemann surfaces

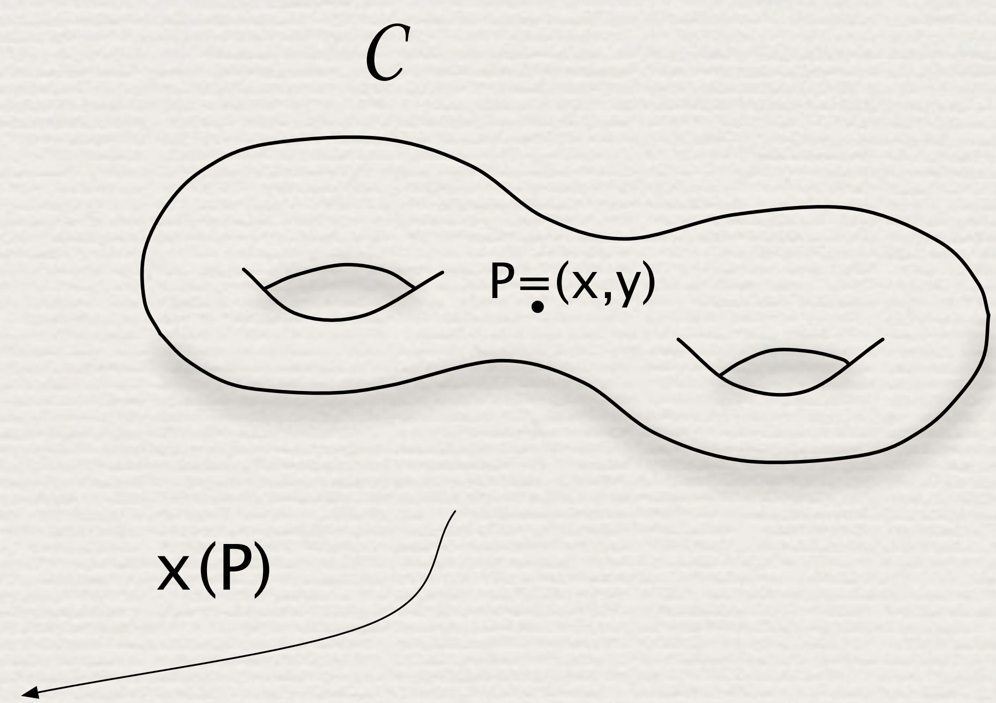
- Definition: A Riemann surface is a connected one-dimensional complex analytic manifold, i.e., a connected two-dimensional real manifold R with a complex structure Σ on it
- Theorem: All compact Riemann surfaces can be described as compactifications of non-singular algebraic curves

Algebraic curves

- Definition: plane algebraic curve C subset in $\mathbb{C}^2$,
  $C = \{(x, y) \in \mathbb{C}^2 | f(x, y) = 0),$

$$f(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} a_{ij} x^i y^j = \sum_{j=0}^{N} a_j(x) y^j$$

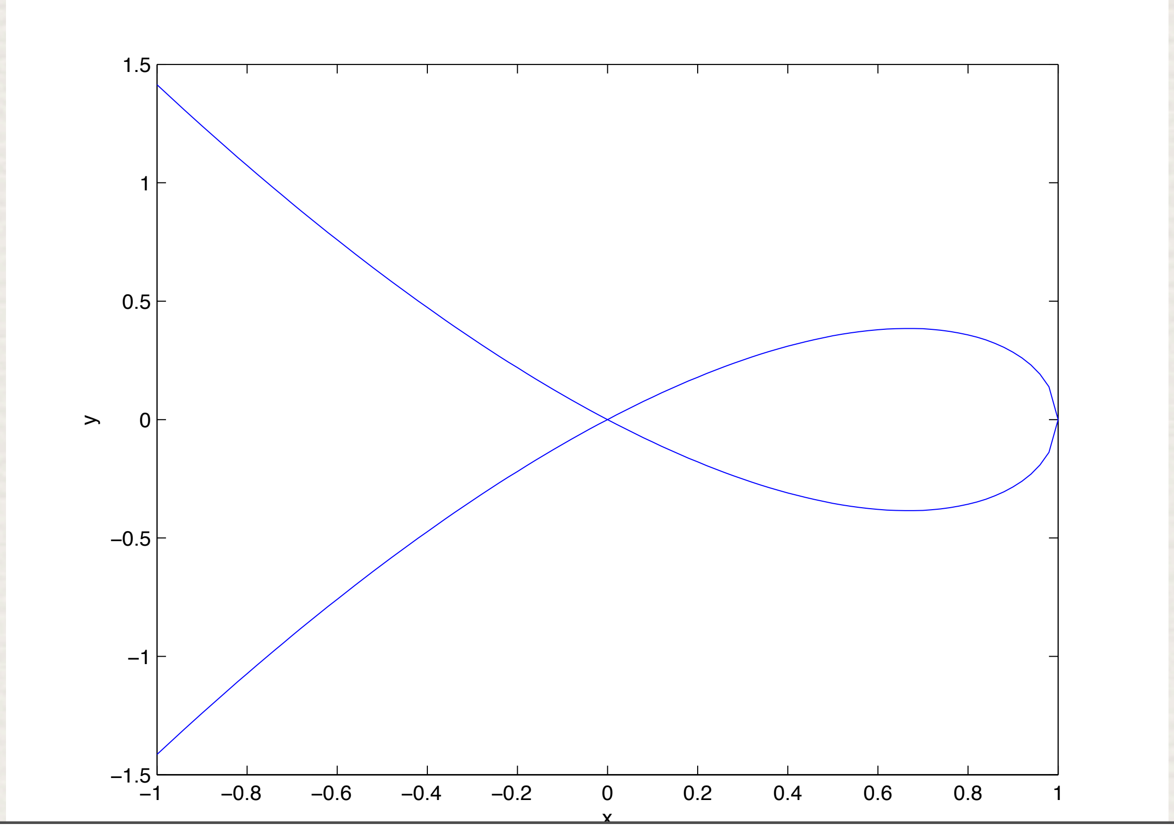




Critical points

- general position: N distinct solutions y_n for given x, N sheets of the Riemann surface
- Implicit function theorem: unique solution to $f(x,y)=0$ in vicinity of solution (x_0,y_0) if $f_y(x_0,y_0)\neq 0$
- branch point: $f(x_0,y_0)=f_y(x_0,y_0)=0$, but $f_x(x_0,y_0)\neq 0$
  singular point: $f(x_0,y_0)=f_y(x_0,y_0)=f_x(x_0,y_0)=0$
- critical points given by the resultant $R(x)$ of $Nf-f_y y$ and f_y

simple double point:
$$y^2+x^3-x^2=0$$



Resultant

- resultant of $Nf - f_y y$ and f_y, $2N \times 2N$ Sylvester determinant

$R(x) =$

$$\begin{pmatrix}
a_{N-1} & 2a_{N-2} & \dots & Na_0 & 0 & \dots & \dots & 0 \\
0 & a_{N-1} & 2a_{N-2} & \dots & Na_0 & 0 & \dots & 0 \\
\vdots & \ddots & & & & & \ddots & \vdots \\
0 & \dots & \dots & 0 & a_{N-1} & 2a_{N-2} & \dots & Na_0 \\
Na_{N-1} & (N-1)a_{N-2} & \dots & a_1 & 0 & \dots & \dots & 0 \\
0 & Na_{N-1} & (N-1)a_{N-2} & \dots & a_1 & 0 & \dots & 0 \\
\vdots & \ddots & & & & & \ddots & \vdots \\
0 & \dots & \dots & 0 & Na_{N-1} & (N-1)a_{N-2} & \dots & a_1
\end{pmatrix}$$

Numerical root finding

- construct companion matrix (has $R(x)$ as the characteristic polynomial), find eigenvalues with machine precision
- multiple zeros are not found with machine precision, ex. $y^7 = x(x-1)^2$ Klein curve, $R(x) = x^6(x-1)^{12}$, `roots(R(x))` returns the following cluster of roots

```
1.1053 + 0.0297i
1.1053 - 0.0297i
1.0736 + 0.0790i
1.0736 - 0.0790i
1.0224 + 0.1032i
1.0224 - 0.1032i
0.9686 + 0.0980i
0.9686 - 0.0980i
0.9264 + 0.0686i
0.9264 - 0.0686i
0.9037 + 0.0245i
0.9037 - 0.0245i,
```

polynomial root finding

- badly conditioned problem
- Zeng: *multroot package* for multiple roots (Newton iteration, minimize error by choice of multiplicity structure)
- resultant high order polynomial, therefore direct Newton iteration in x and y. Initial iterates from resultant with respect to x and y, pairing
- postprocessing for higher order zeros

Singularities

- multiple roots are tested for vanishing $f_x(x,y)$
- infinity: homogeneous coordinates X,Y,Z via $x = X/Z$, $y = Y/Z$

$$F(X,Y,Z) = Z^d f(X/Z, Y/Z) = 0$$

  infinite points: $Z = 0$, finite points: $Z = 1$

- Singular points at infinity: $F_X(X,Y,0) = F_Y(X,Y,0) = F_Z(X,Y,0) = 0$

Example

- curve

$$f(x, y) = y^3 + 2x^3y - x^7 = 0 \; ,$$

- finite branch points

```
bpoints =
       -0.3197 - 0.9839i
        0.8370 - 0.6081i
       -1.0346
        0
        0.8370 + 0.6081i
       -0.3197 + 0.9839i
```

- singularities,

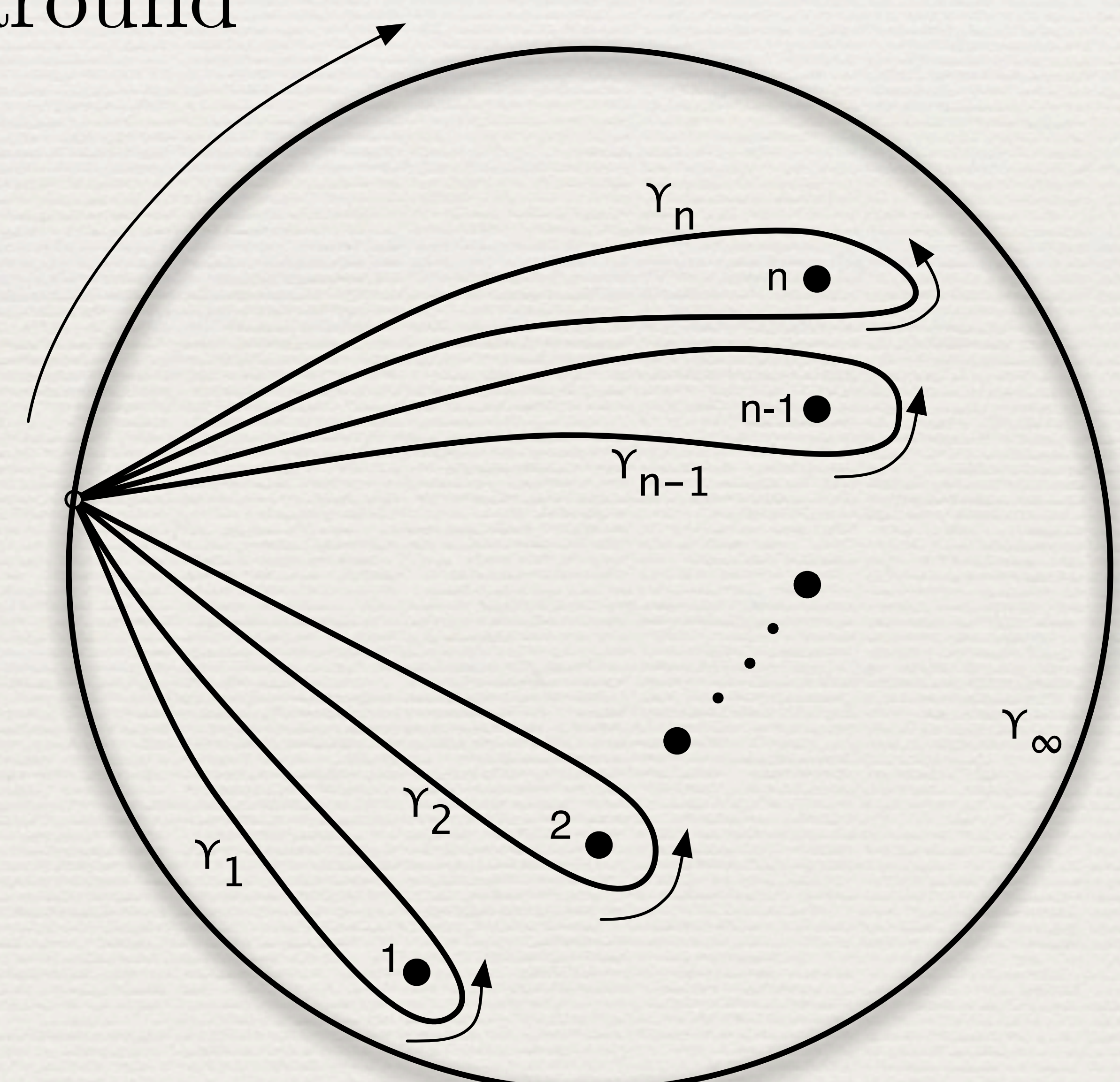
sing =	X	Y	Z	
	0	0	1	4
	0	1	0	9

corresponding to $x = y = 0$ and $Y = 1$, $X = Z = 0$

Fundamental group

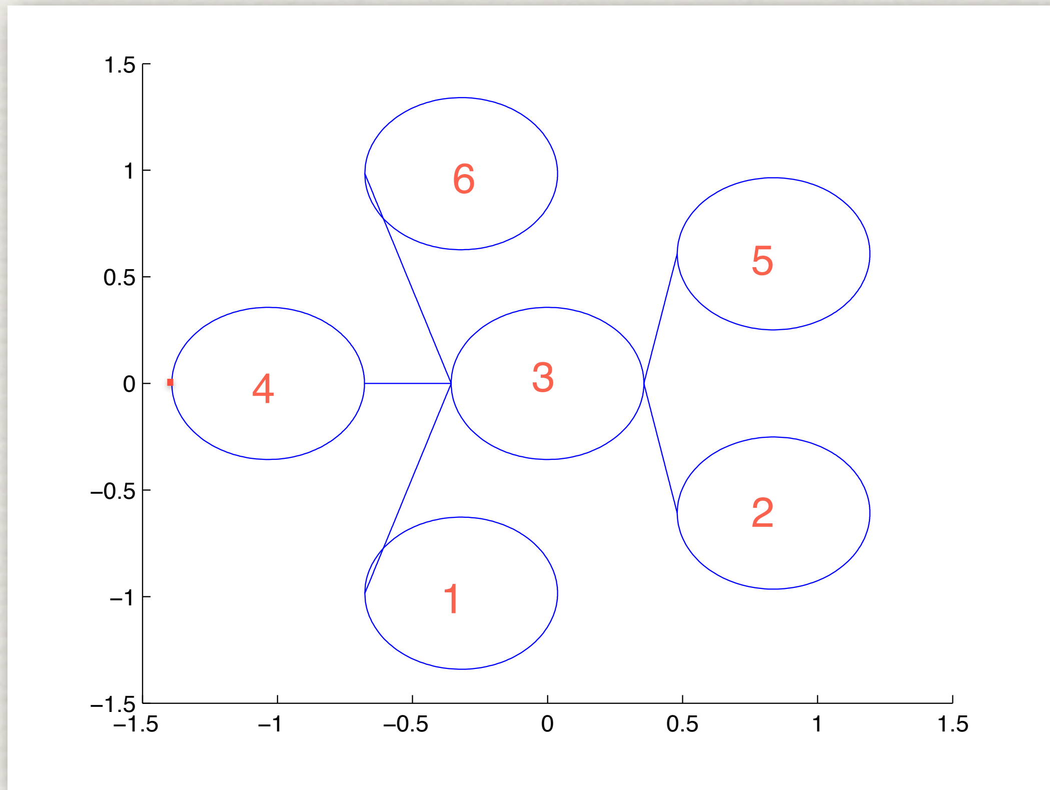
- branching structure at critical points, lift closed contours in the base around points $b_1, \ldots, b_n$ to the covering
- generators $\{\gamma_k\}_{k=1}^n$ of fundamental group $\pi_1(\mathbb{CP}^1 \backslash \{b_1, \ldots, b_n\})$

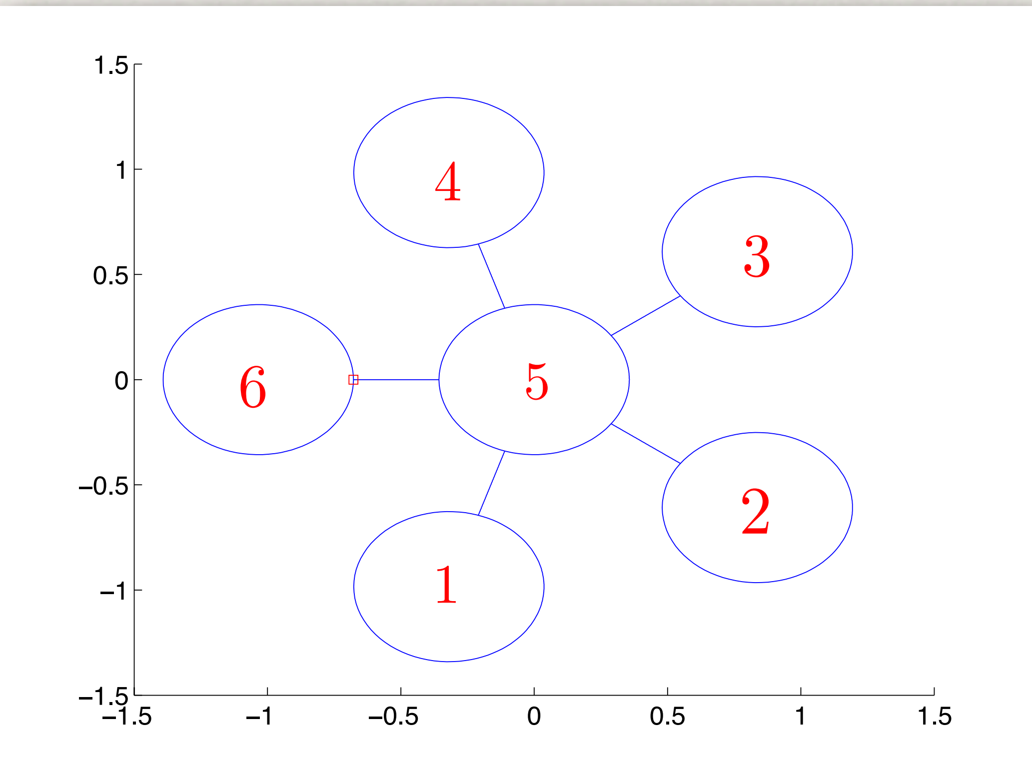
  $\gamma_1 \gamma_2 \cdots \gamma_n \gamma_\infty = \text{id}$



Minimal spanning tree

- Maple: halfcircles around critical points, deformation of connecting paths
- shortest integration paths: start with critical point close to the base, choose point with minimal distance, iterate (Frauendiener, K, Shramchenko 2011)
- ex:



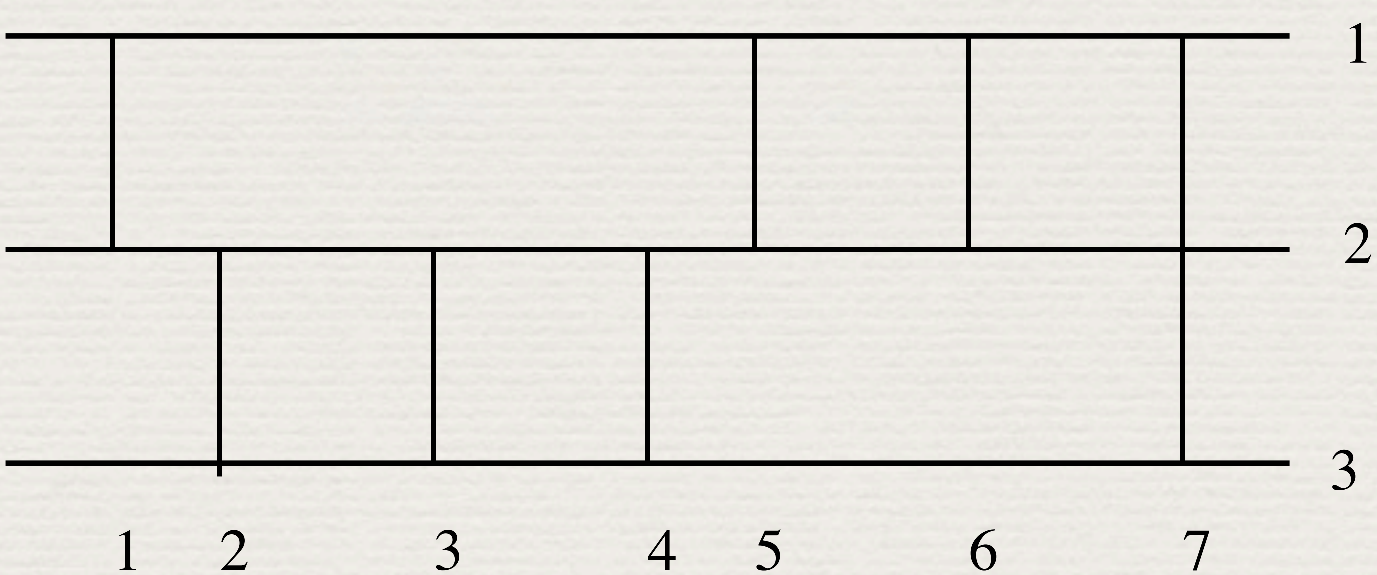


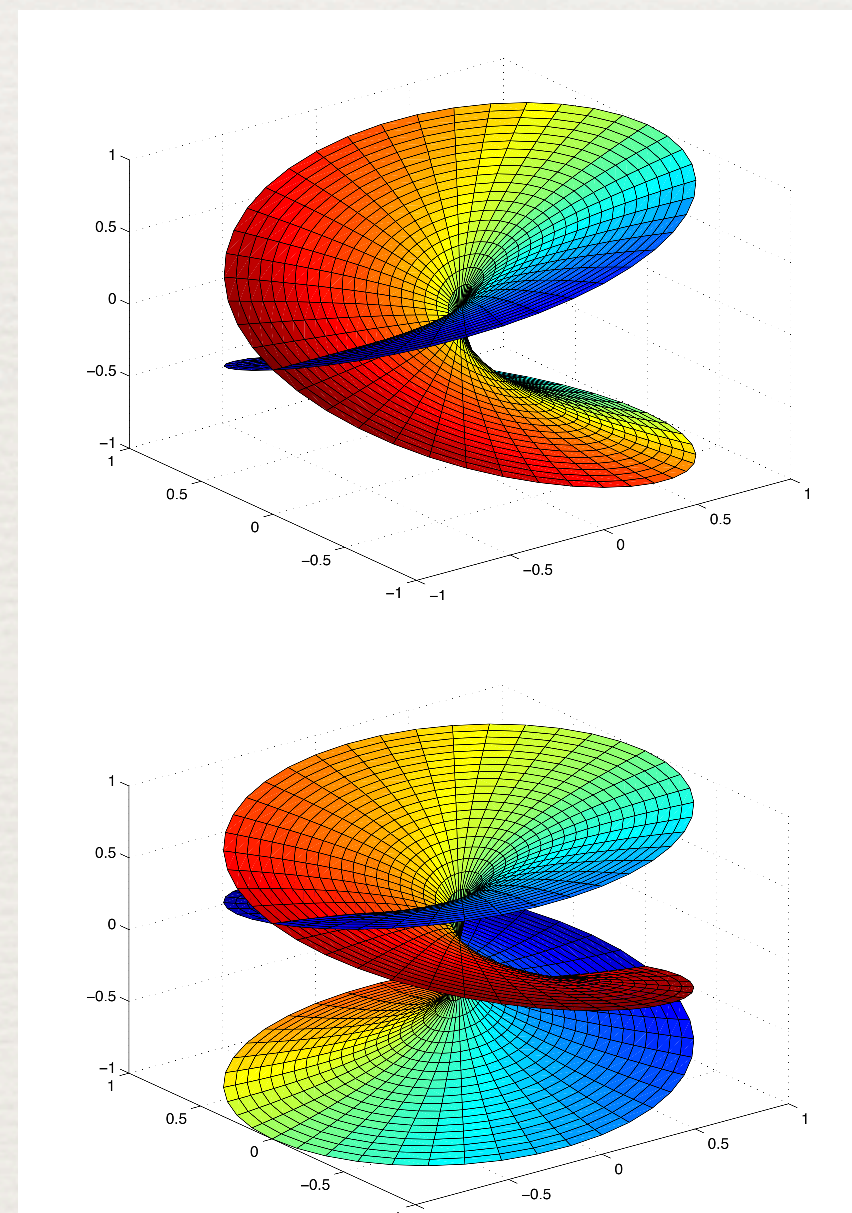
Monodromies

- analytic continuation along a generator: sheets can change
- monodromy at infinity follows from condition on generators

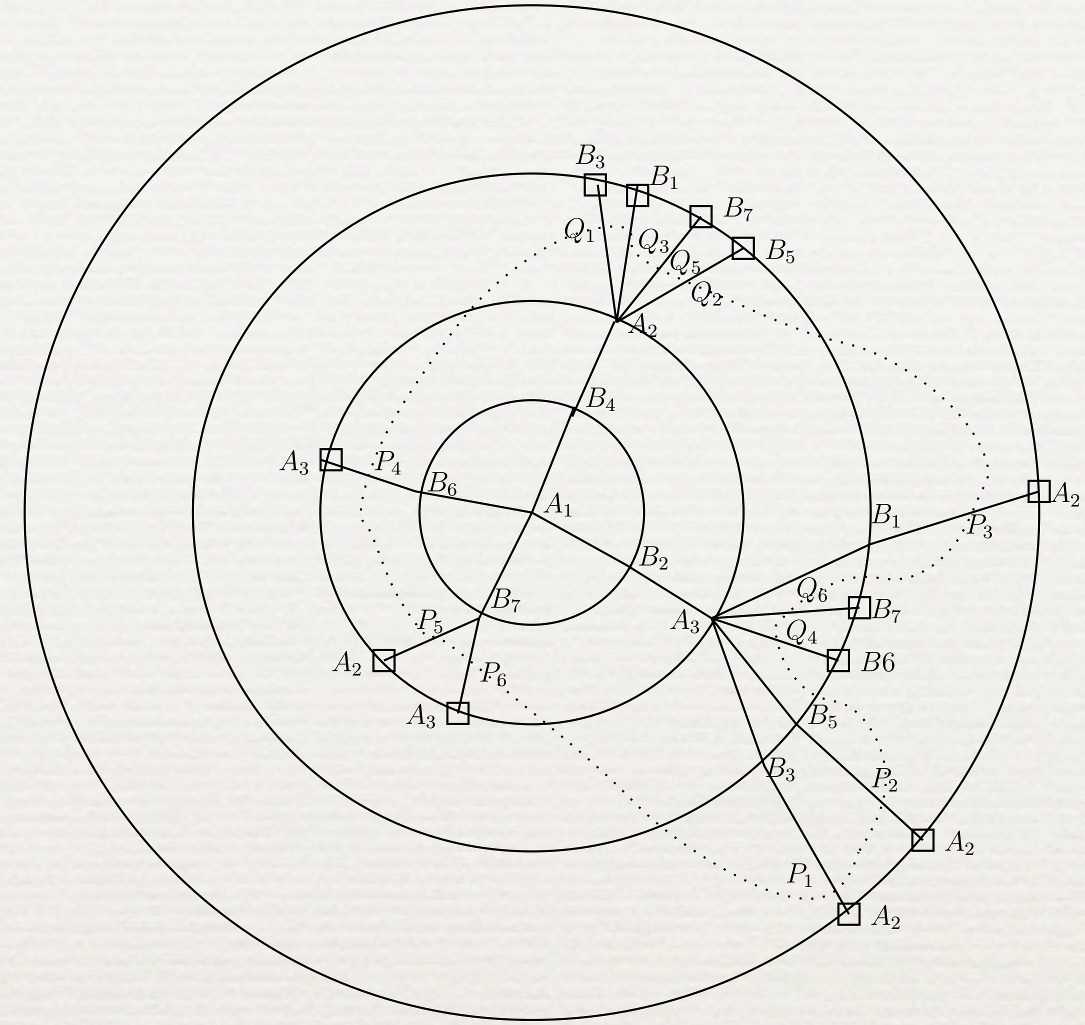
- ex.:

```
Mon =

 2    1    1    1    2    2    2
 1    3    3    3    1    1    3
 3    2    2    2    3    3    1
```





Homology



- ex.:

```
Mon =
          1     3     1     2     1     3     2
          3     2     3     1     3     2     3
          2     1     2     3     2     1     1.
```

- Tretkoff-Tretkoff algorithm: Riemann surface connected, planar tree for given monodromies
- *2g+N-1* closed contours built from the generators of the fundamental group, with known intersection numbers
- canonical basis of the homology:

$$a_i \circ b_j = -b_j \circ a_i = \delta_{ij} \quad i, j = 1, \ldots, g$$

Canonical cycles

- canonical basis of the homology :

$$a_i = \sum_{j=1}^{2g+N-1} \alpha_{ij} c_j \ , \quad b_i = \sum_{j=1}^{2g+N-1} \alpha_{i+g,j} c_j \ , \quad i = 1, \ldots, g$$

  c_j: $2g + N - 1$ closed contours from the planar graph

- remaining cycles homologous to zero (test for numerical accuracy)

$$0 = \sum_{j=1}^{2g+N-1} \alpha_{ij} c_j \ , \quad i = 2g+1, \ldots, 2g+N-1$$

- ex.:

```
acycle{1} =
          1      4      2      3      3      2      1
acycle{2} =
          1      4      2      1      3      2      1
bcycle{1} =
          1      6      3      2      1
bcycle{2} =
          2      3      3      2      3      5      2,
```

Puiseux expansion

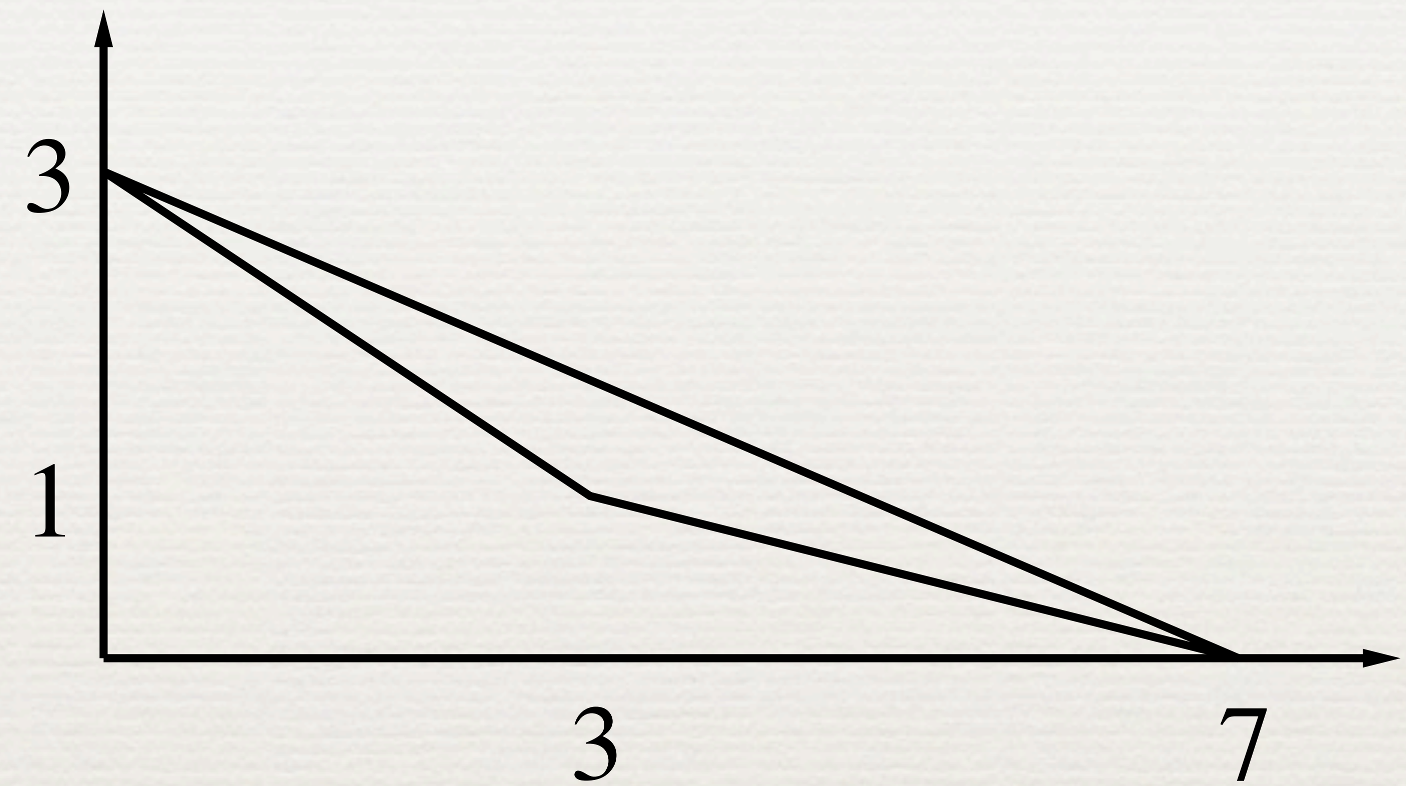
- desingularization: atlas of local coordinates to identify all sheets in the vicinity of the singularity
- $y^2 = x$, no Taylor expansion $y(x)$ near $(0,0)$, Puiseux expansion

$$x = t^r \ , \quad y = \alpha_1 t^{s_1} + ...$$

$r, s_1, \ldots \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \ldots$

- $y = 0$ zero of order m for $f(0, y) = 0$, m inequivalent expansions needed to identify all sheets, *singular part*

Newton polygon



- ex.: $f(x,y) = y^3 + 2x^3y - x^7 = 0$

```
PuiExp{1} =
      2.0000          3.0000              0 + 1.4142i
      2.0000          3.0000              0 - 1.4142i
      1.0000          4.0000              0.5000
PuiExp{2} =
      4.0000          7.0000             -1.0000
      4.0000          7.0000              0 + 1.0000i
      4.0000          7.0000              0 - 1.0000i
      4.0000          7.0000              1.0000
```

`PuiExp{1}` for $(0,0)$ ($[0,0,1]$), `PuiExp{2}` for infinity ($[0,1,0]$)

Holomorphic 1-forms

- holomorphic in each coordinate chart, g-dimensional space
- Noether:

$$\omega_k = \frac{P_k(x,y)}{f_y(x,y)} dx \ ,$$

adjoint polynomials $P_k(x,y) = \sum_{i+j \leq d-3} c_{ij}^{(k)} x^i y^j$,
degree at most $d-3$ in x and y ($d = \max(i+j)$ for $a_{ij} \neq 0$)

- singular point P: δ_P conditions via Puiseux expansions
- infinity: homogeneous coordinates
- ex.: $f(x,y) = y^3 + 2x^3 y - x^7 = 0$

$$\omega_1 = \frac{x^3}{3y^2+2x^3}, \quad \omega_2 = \frac{xy}{3y^2+2x^3}$$

Cauchy integral approach

- numerical problem: cancellation errors, ex. $\frac{e^x-1}{x}$ for $x \to 0$
- Cauchy formula

$$f(t) = \frac{1}{2\pi i}\int_\gamma \frac{f(t')}{t'-t}dt'$$

- closed contours around critical points identified via monodromy group
- series in t for holomorphic f $(|t| < |t'|)$

$$f(t) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} t^n \int_\gamma f(t')\frac{dt'}{(t')^{n+1}}$$

- Puiseux series: $f = y$;
  holomorphicity condition for differentials (no negative powers)
- infinity: express γ_∞ in terms of the γ_i

- infinity: homogeneous coordinates
- singular part of more than one term, additional conditions
- ex.: $f(x, y) = x^3 + 2x^3y - x^7 = 0$

```
c{1} =
          0         0         0         0         0
          0         0         0         0         1
          0         0         0         0         0
          0         0         0         0         0
          0         0         0         0         0
c{2} =
          0         0         0         0         0
          0         0         0         0         0
          0         0         0         0         0
          0         0         0         1         0
          0         0         0         0         0
```

polynomials $P_1 = x^3$ and $P_2 = xy$

Numerical integration

- Gauss-Legendre integration: expansion of integrand in terms of Legendre polynomials $\mathcal{F}(x_l) = \sum_{k=0}^{N_l} a_k \mathcal{P}_k(x_l)$, $l = 0, \ldots, N_l$

$$\int_{-1}^{1} \mathcal{F}(x)dx \sim \sum_{k=0}^{N_l} a_k \int_{-1}^{1} \mathcal{P}_k(x)dx$$

- integration:

$$\int_{-1}^{1} \mathcal{F}(x)dx \sim \sum_{k=0}^{N_l} \mathcal{F}(x_k)\mathcal{L}_k$$

- analytic continuation of y_j along the γ_i on the collocation points x_l, integration of the holomorphic differentials

Riemann matrix

- a- and b-periods

$$\mathbf{B} = \mathcal{A}^{-1}\mathcal{B}$$

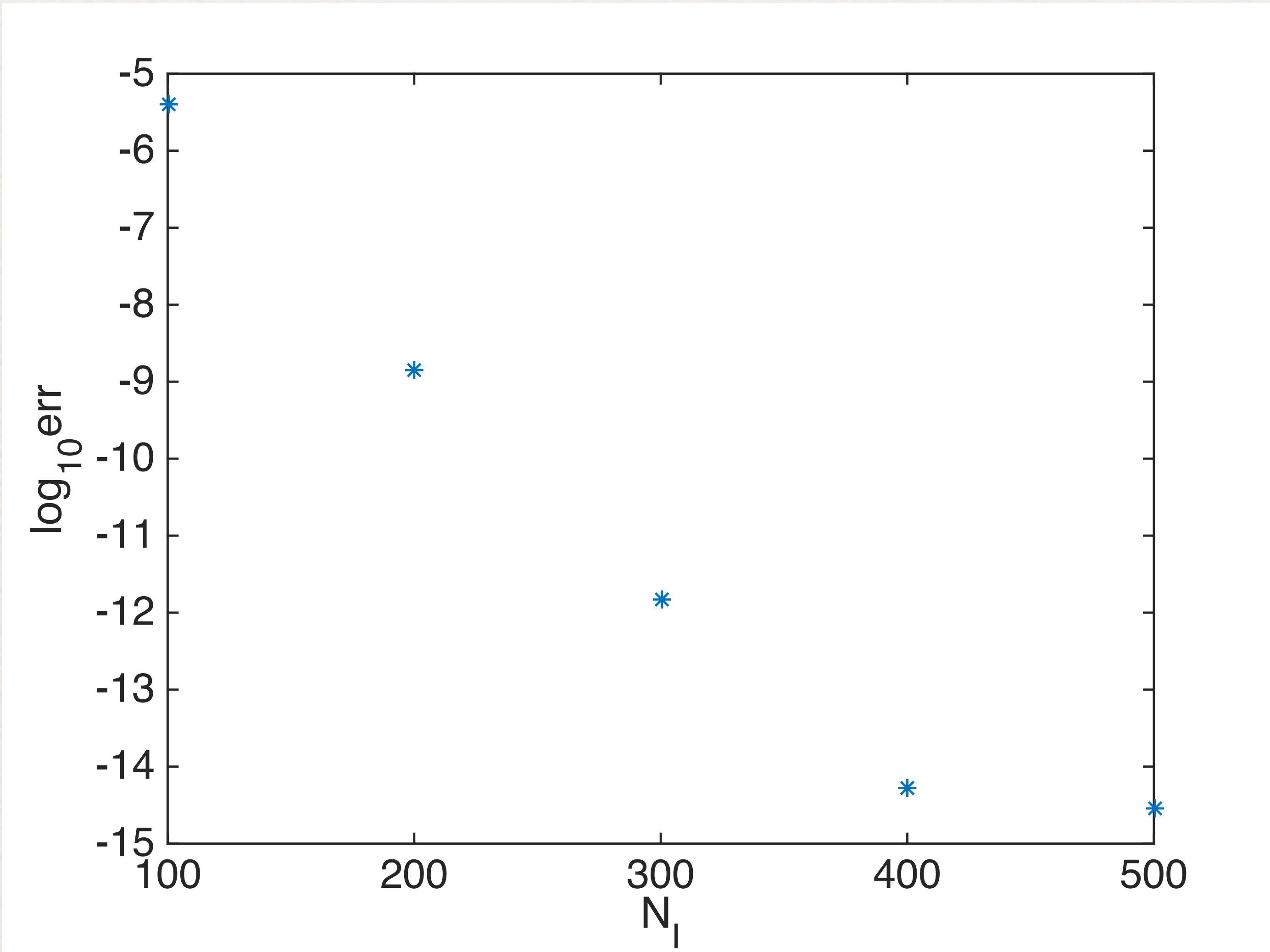
- numerical asymmetry of Riemann matrix as test
- ex.:

```
RieMat =
   0.3090 + 0.9511i   0.5000 - 0.3633i
   0.5000 - 0.3633i  -0.3090 + 0.9511i.
```

Performance

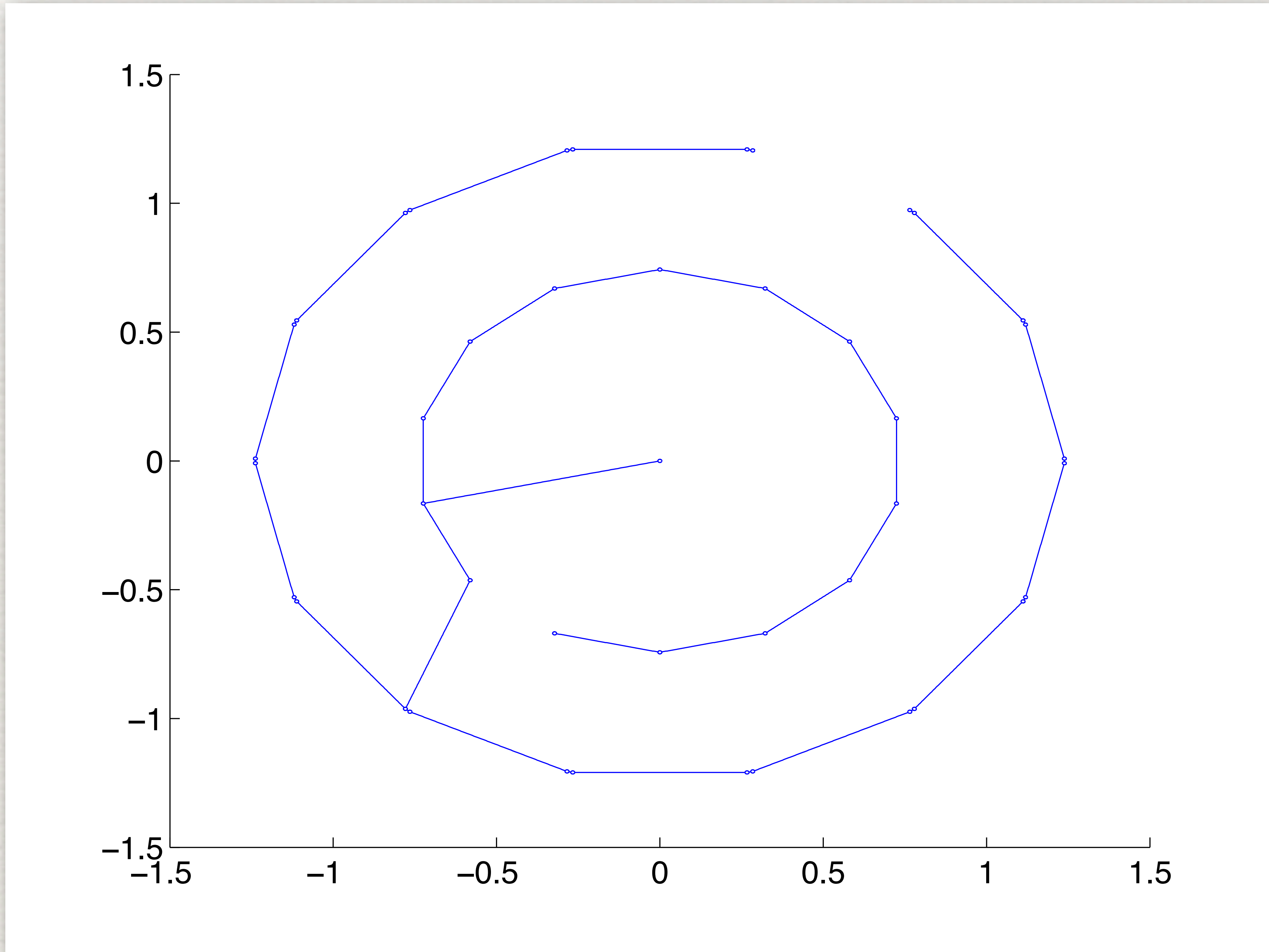
- error: asymmetry of the Riemann matrix and periods of cycles homologous to 0 for

$$f(x,y) = y^9 + 2x^2y^6 + 2x^4y^3 + x^6 + y^2 = 0$$



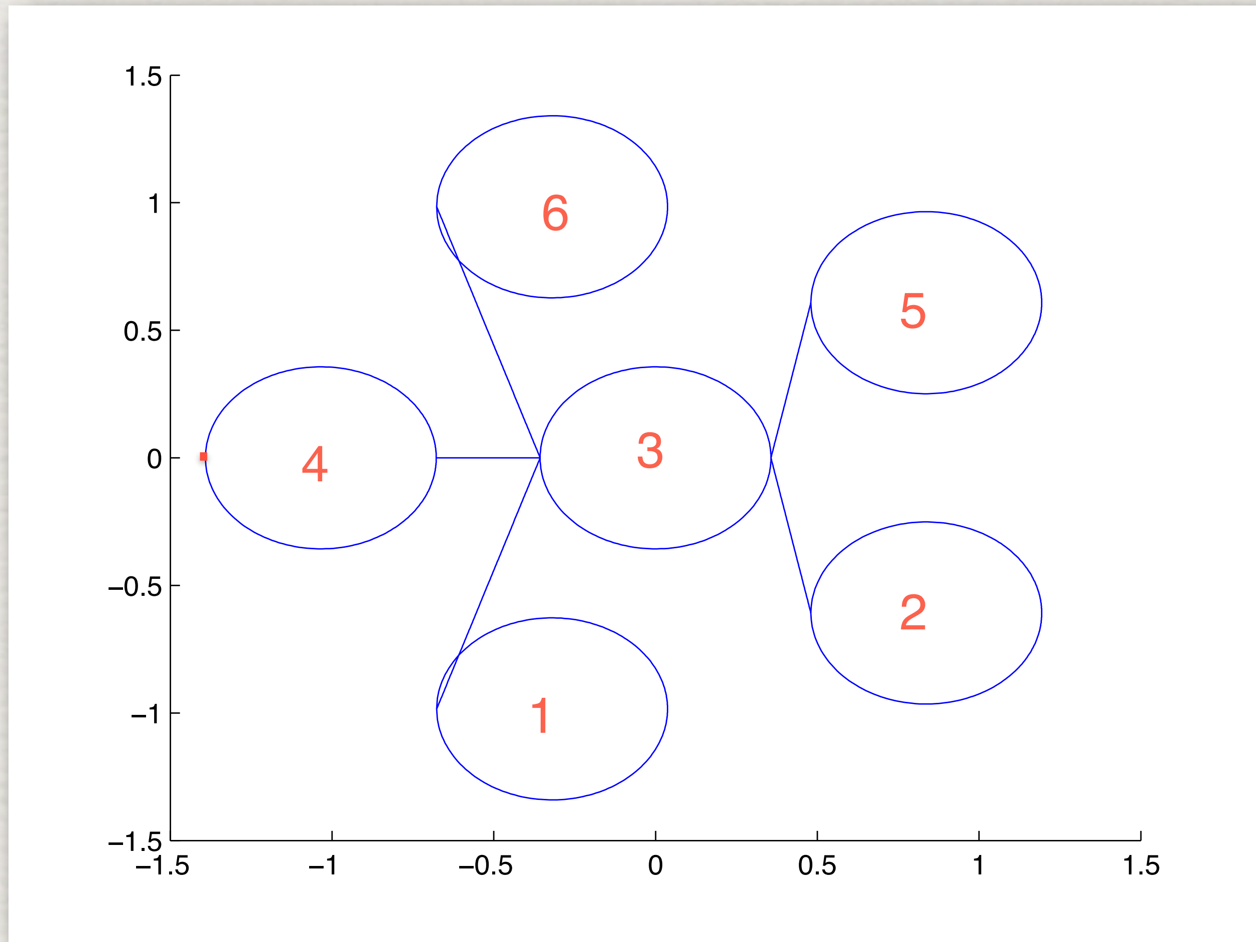
- $$f(x, y) = y^9 + 2x^2y^6 + 2x^4y^3 + x^6 + y^2 = 0$$

genus 16, 42 finite branch points, two singular points $(0, 0, 1)$ and (1,0,0), minimal distance between branch points 0.018



Abel map

- $A(P)$: determine closest marked point to P, analytic continuation of y from there and integration as before.
- critical points, infinity: Cauchy formula



Real Riemann surfaces

- in applications, solutions to PDEs in terms of theta functions must satisfy reality and smoothness conditions
- real Riemann surfaces: anti-holomorphic involution, convenient form of the homology basis
- smoothness: study of the theta divisor (zeros of the theta function) (Dubrovin, Natanzon, Vinnikov)

Theta functions

- theta series

$$\Theta_{\mathrm{pq}}(\mathrm{z}, \mathbb{B}) = \sum_{\mathrm{N} \in \mathbb{Z}^g} \exp\left\{ \mathrm{i}\pi \left\langle \mathbb{B}\left(\mathrm{N}+\mathrm{p}\right), \mathrm{N}+\mathrm{p} \right\rangle + 2\pi\mathrm{i} \left\langle \mathrm{z}+\mathrm{q}, \mathrm{N}+\mathrm{p} \right\rangle \right\}$$

- Deconinck, B., Heil, M., Bobenko, A., van Hoeij, M., Schmies, M.: Computing Riemann theta functions. Mathematics of Computation, **73**, 1417–1442 (2004)

- approximated as a sum $|N_i| \leq \mathcal{N}_\epsilon$, $i = 1, \ldots, g$

$$\mathcal{N}_\epsilon > \sqrt{-\frac{\ln \epsilon}{\pi y_{min}}} + \frac{1}{2}$$

Symplectic transformation

- $$\mathcal{A}_g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

  A, B, C, D $g \times g$ integer matrices

  $$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}; \tag{2}$$

- Riemann matrix

  $$\mathbb{H}^g \mapsto \mathbb{H}^g : \quad \mathbb{B} \mapsto \tilde{\mathbb{B}} = (A\mathbb{B} + B)(C\mathbb{B} + D)^{-1}. \tag{3}$$

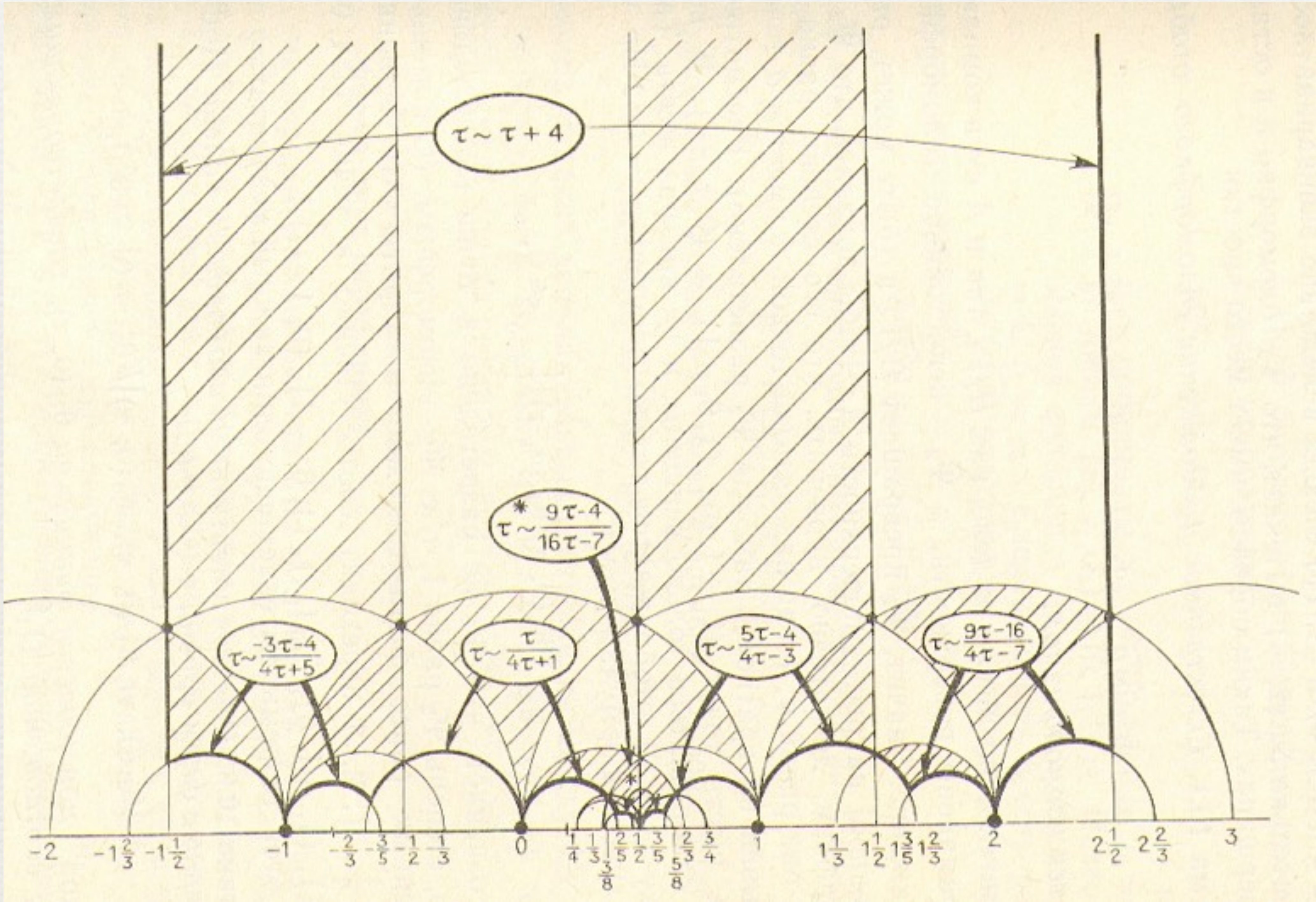
- theta function

  $$\Theta_{\tilde{\mathrm{p}}\tilde{\mathrm{q}}}(\mathcal{M}^{-1}\mathrm{z}, \tilde{\mathbb{B}}) = k\sqrt{\det(\mathcal{M})} \exp\left(\frac{1}{2}\sum_{i \leq j} z_i z_j \frac{\partial}{\partial \mathbb{B}_{ij}} \ln\det\mathcal{M}\right) \Theta_{\mathrm{pq}}(\mathrm{z}), \tag{4}$$

  $$\mathcal{M} = C\mathbb{B} + D, \quad \begin{pmatrix} \tilde{\mathrm{p}} \\ \tilde{\mathrm{q}} \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \mathrm{p} \\ \mathrm{q} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathrm{diag}(CD^T) \\ \mathrm{diag}(AB^T) \end{pmatrix}, \tag{5}$$

Fundamental domain

$$|\Re\sigma| \leq \frac{1}{2}, \Im\sigma > \frac{1}{2}\sqrt{3}$$



Mumford: Tata Lectures on Theta Functions

Siegel's fundamental domain

- subset of $\mathbb{H}^g$ such that $\mathbb{B} = X + \mathrm{i}Y \in \mathbb{H}^g$ satisfies:

  1. $|X_{nm}| \leq 1/2$, $n, m = 1, \ldots, g$,
  2. Y is in the fundamental region of Minkowski reductions,
  3. $|\det(C\mathbb{B} + D)| \geq 1$ for all C, D.

- *quasi-inversion*: $\mathbb{B} \mapsto -\mathbb{B}/\mathbb{B}_{11}$

$$A = \begin{pmatrix} 0 & \mathbf{0}_{g-1}^T \\ \mathbf{0}_{g-1} & \mathbf{1}_{g-1,g-1} \end{pmatrix}, \quad B = \begin{pmatrix} -1 & \mathbf{0}_{g-1}^T \\ \mathbf{0}_{g-1} & \mathbf{0}_{g-1,g-1} \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & \mathbf{0}_{g-1}^T \\ \mathbf{0}_{g-1} & \mathbf{0}_{g-1,g-1} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \mathbf{0}_{g-1}^T \\ \mathbf{0}_{g-1} & \mathbf{1}_{g-1,g-1} \end{pmatrix}, \tag{1}$$

Lattice reduction

- Lattice generated by all integer combinations of the vectors t_i of T, $\Im\mathbb{B} = T^T T$
- Minkowski: shortest lattice vectors which can be extended to a basis of $\mathcal{L}$.
- Gram-Schmidt vectors

$$t_i^* = t_i - \sum_{j=1}^{i-1} \mu_{i,j} t_j^*, \quad \mu_{i,k} = \frac{\langle t_i, t_k^* \rangle}{||t_i^*||^2}$$

- LLL condition

$$||t_k^*||^2 \geq (\delta - \mu_{k,k-1}^2)||t_{k-1}^*||^2 \tag{1}$$

Example

- random matrix $Y = \Im\mathbb{B}$

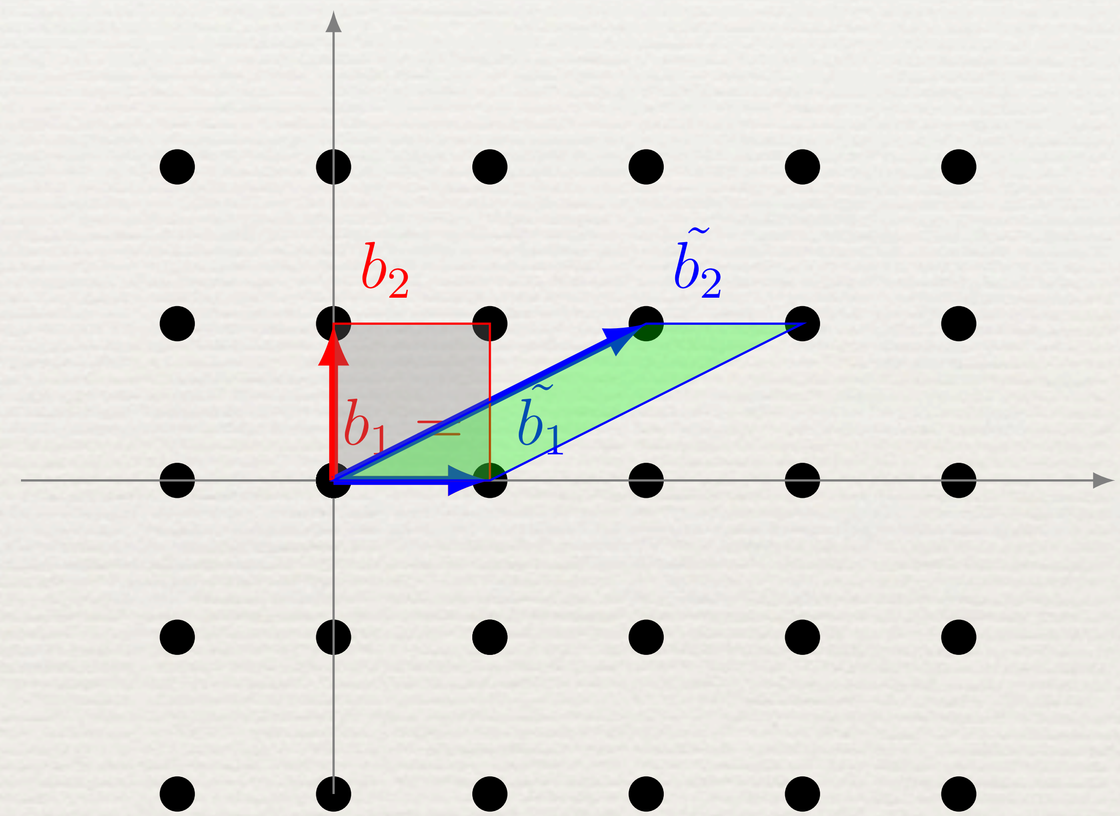
```
Y =

1.7472    0.5191    1.0260    0.6713
0.5191    1.3471    0.2216   -0.5122
1.0260    0.2216    0.6801    0.4419
0.6713   -0.5122    0.4419    0.7246.
```

- Minkowski reduction

```
0.2205    0.0443    0.0342    0.0351
0.0443    0.3636    0.1660   -0.0294
0.0342    0.1660    0.3688    0.1516
0.0351   -0.0294    0.1516    0.3753.
```

- LLL reduced matrix ($\delta = 3/4$)

```
 0.3753    0.0294   -0.1516    0.0351
 0.0294    0.3636    0.1660   -0.0443
-0.1516    0.1660    0.3688   -0.0342
 0.0351   -0.0443   -0.0342    0.2205.
```



C. Jaber (thesis, 2017)

Davey-Stewartson equations

$$\mathrm{i}\,\psi_t + \psi_{xx} - \alpha^2\,\psi_{yy} + 2\,(\Phi + \rho\,|\psi|^2)\,\psi = 0,$$

$$\alpha = \mathrm{i}, 1, \quad \rho = \pm 1, \quad \Phi_{xx} + \alpha^2\,\Phi_{yy} + 2\rho\,|\psi|^2_{xx} = 0,$$

- model the evolution of weakly nonlinear water waves traveling predominantly in one direction, wave amplitude slowly modulated in two horizontal directions, plasma physics, ...
- completely integrable, theta-functional solutions (Malanyuk 1994, Kalla 2011)
- algorithm to transform computed homology basis to `Vinnikov' basis (K, Kalla 2011)

Trott curve

- M-curve, $g = 3$, real simple branch points $(s = (1, -1, -1))$

$$144\,(x^4 + y^4) - 225\,(x^2 + y^2) + 350\,x^2y^2 + 81 = 0$$

$\mathrm{DS1}^+$, $\lambda(a) = -0.2$, $\lambda(b) = 0.2$ $\qquad \alpha = \mathrm{i},\ \rho = 1$

$t \in [-2, 2]$

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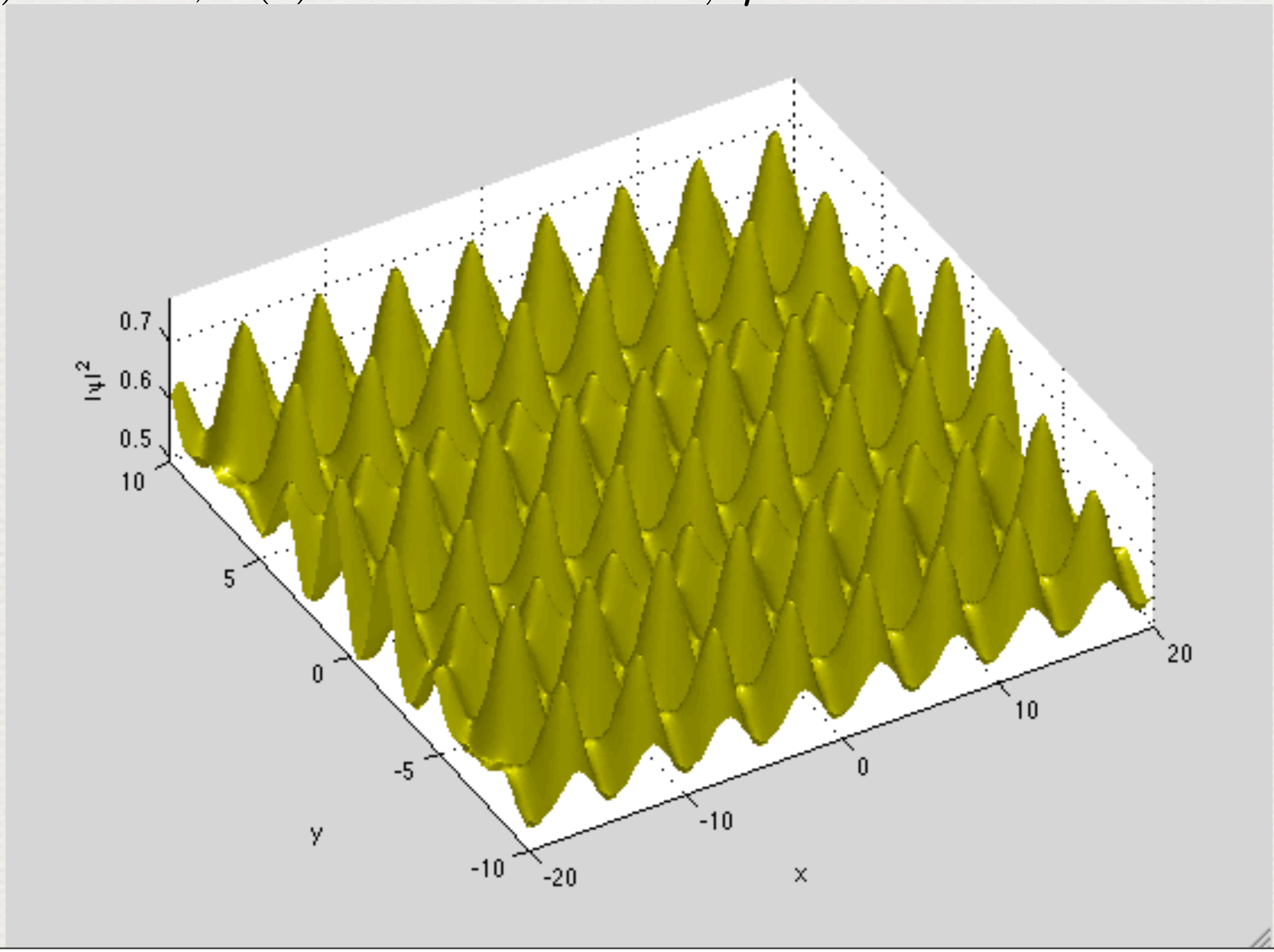
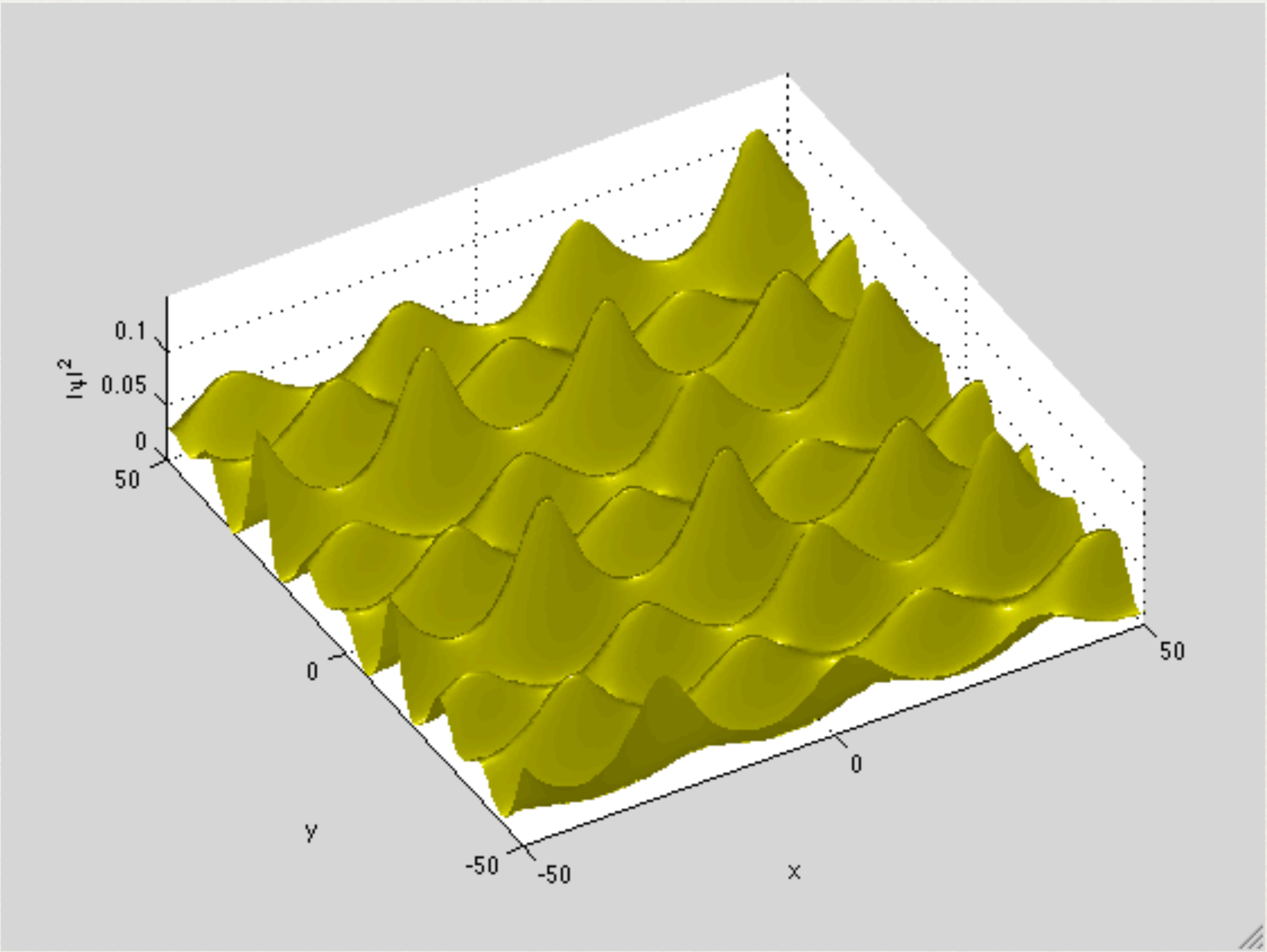


DS on Fermat curve g=3

DS2$-$, $\lambda(a) = -1.5 + \mathrm{i}$, $\lambda(a) = -1.5 - \mathrm{i}$ $\quad t \in [-5, 5]$

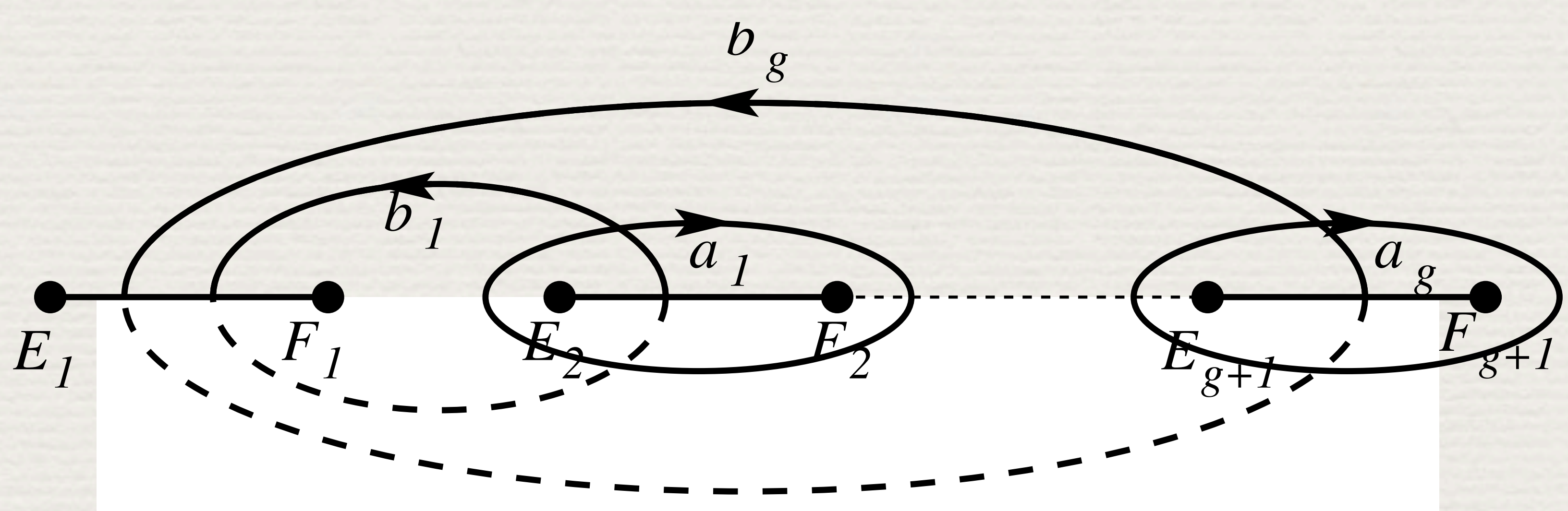
DS on Fermat curve g=3

DS2−, $\lambda(a) = -1.5 + \mathrm{i}$, $\lambda(a) = -1.5 - \mathrm{i}$ $\quad t \in [-5, 5]$



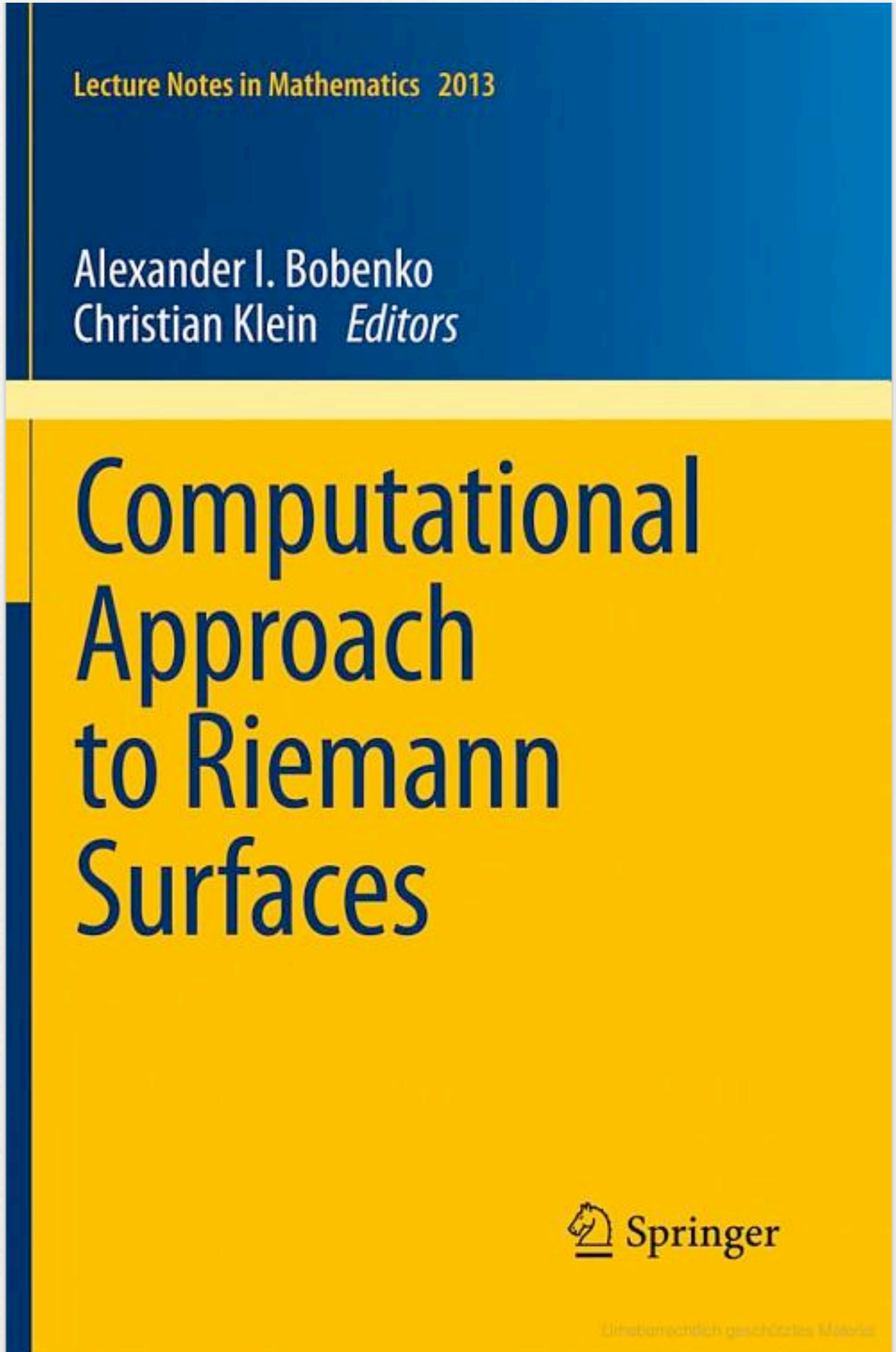
Hyperelliptic surfaces

- general surface: analytic continuation most time consuming
- y: square root of polynomial in x, holomorphic differentials known, $y^2 + \prod_{i=1}^{2g+2}(x - x_i) = 0$
- analytic continuation of the root trivial (square root, correct unwanted sign changes)
- branch points prescribed, can almost collapse
- homology can be chosen a priori



Outlook

- more efficient determination of critical points, homotopy tracing, endgame
- Siegel transformation of the Riemann matrix to fundamental domain in genus 3
- parallelization of theta functions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