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Analysis of Wishart matrices:

Riesz and Wishart laws on graphical cones

Letac-Massam conjecture

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[Faraut, Koranyi, *Analysis on Symmetric Cones, Oxford Press, 1994*], Chapter VII "The Gamma function of a symmetric cone":

"Here begins the serious study of analysis on symmetric cones"

$\Omega \in V = \mathbb{R}^n$: proper $(\bar{\Omega} \cap (-\bar{\Omega}) = \{0\})$ open convex cone

Ω^*: open dual cone $=\{y \in V | \ (x, y) > 0 \ \forall x \in \bar{\Omega} \setminus \{0\}\}$

Characteristic function of a cone

$$\varphi_\Omega(x) = \int_{\Omega^*} e^{-(x,y)} dy = \mathcal{L}_{(\Omega^*, Leb)}(Leb_{\Omega^*})(x)$$

Properties of φ_Ω: If $g \in GL(V)$ is an automorphism of G (i.e. $g\Omega = \Omega$), we have

$$\varphi_\Omega(gx) = |\det g|^{-1} \varphi_\Omega(x).$$

Consequently, $\varphi_\Omega(x)dx$ is the invariant measure of the cone Ω:

$$\int_\Omega f(gx)\varphi_\Omega(x)dx = \int_\Omega f(x)\varphi_\Omega(x)dx.$$

Example $\Omega = \mathbb{R}^+$.

It is

a self-dual cone: $\Omega^* = \mathbb{R}^+$

a homogeneous cone:

$\forall x, y > 0 \exists c \in Aut(\Omega) = \mathbb{R}^+ \quad y = cx.$

Self-dual homogeneous cones are called **symmetric cones**.

Characteristic function and invariant measure of $\mathbb{R}^+$:

$$\varphi_{\mathbb{R}^+}(x) = \int_0^\infty e^{-xy} dy = \frac{1}{x}, \quad x > 0$$

$$\int_0^\infty f(cx) \frac{1}{x} dx = \int_0^\infty f(x) \frac{1}{x} dx, \quad c > 0$$

Gamma function. For $s > 0$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx = \int_0^\infty e^{-x} x^s \varphi_{\mathbb{R}+}(x) dx$$

Gamma integral. For $s > 0$

$$\mathcal{L}_{\mathbb{R}+,\frac{dx}{x}}(x^s)(y) = \mathcal{L}_{\mathbb{R}+}(x^{s-1})(y) = \int_0^\infty e^{-xy} x^{s-1} dx = \Gamma(s) y^{-s}$$

The Riesz distributions R_s on $\mathbb{R}^+$ are defined by

$$\mathcal{L}(R_s)(y) = y^{-s} = \text{ a power function}$$

Riesz distributions on $\mathbb{R}^+$ are positive measures if and only if $s > 0$. Then they have density $R_s(x) = x^{s-1}/\Gamma(s)$.

Gamma integrals are important in statistics:

$$\text{the functions}: \quad x \mapsto \frac{e^{-xy}}{\mathcal{L}(R_s)(y)} R_s(x) =: \gamma_{s,y}(x)$$

are probability densities for $s, y > 0$.

They are GAMMA densities on $\mathbb{R}^+$ (interpolation of χ_n^2)

Their Laplace transform: $\mathcal{L}(\gamma_{s,y})(z) = (1 + zy^{-1})^{-s}$.

If μ is a measure on a cone $\Omega \subset V = \mathbb{R}^n$, then the family of probability measures

$$\gamma_y(dx) = \frac{e^{-(x,y)}}{\mathcal{L}(\mu)(y)} \mu(dx)$$

is called exponential family generated by μ.

Cone of positive definite symmetric matrices $S_n = Sym^+(n, \mathbb{R})$

Crucial in multivariate statistics.

Generalized power function of matrix argument $x \in S_n$

$$\Delta_{\underline{s}}(y) = \prod_{i=1}^{n} \left(\frac{\det y_{\leq i}}{\det y_{<i}} \right)^{s_i} \quad \text{"past power function"}$$

If $y = diag(y_1, \ldots, y_n)$, we have $\Delta_{\underline{s}}(y) = \prod_{i=1}^{n} y_i^{s_i}$

For constant $\underline{s} = s(1, \ldots, 1)$, we have $\Delta_{\underline{s}}(y) = (\det y)^s$

Gamma integrals on S_n: Siegel integrals(1935, number theory), appeared before in statistics(Wishart 1928), computed by Ingham (1933).

Characteristic function and invariant measure density $\varphi_{S_n}(x) = (\det x)^{-\frac{n+1}{2}}$

Gamma function of S_n: for $s_j > \frac{j-1}{2}$ and $c_n = (2\pi)^{\frac{n(n-1)}{4}}$

$$\Gamma_{S_n}(\underline{s}) = \int_{S_n^+} e^{-tr(x)} \Delta_{\underline{s}}(x) \varphi_{S_n}(x) dx = c_n \prod_i \Gamma(s_j - \frac{j-1}{2})$$

Gamma-Siegel integral

$$\int_{S_n^+} e^{-tr(xy)} \Delta_{\underline{s}}(x)\varphi_{S_n}(x)dx = \Gamma_{S_n}(\underline{s})\Delta_{\underline{s}}(y^{-1}) = \Gamma_{S_n}(\underline{s})\delta_{-\underline{s}}(y)$$

where $s_j > \frac{j-1}{2}$ and $\delta_{\underline{s}}(y)$ is the "future power function":

$$\delta_{\underline{s}}(y) = \prod_{i=1}^{n} \left(\frac{\det y_{\geq i}}{\det y_{>i}} \right)^{s_i}.$$

A.c. Riesz measures $R_{\underline{s}}(x) = \Delta_{\underline{s}}(x)\varphi_{S_n}(x)/\Gamma_{S_n}(\underline{s})$ have Laplace transform $\Delta_{\underline{s}}(y^{-1}) = \delta_{-\underline{s}}(y)$.
(*There exist also singular positive Riesz measures*)

Exp. families of Riesz measures: Wishart measures $\gamma_{\underline{s},y}$

The parameter $\underline{s}$ is called the shape parameter, y is the scale parameter

The density of $\gamma_{\underline{s},y}$:

$$e^{-tr(xy)}\frac{\Delta_{\underline{s}}(x)\varphi_{S_n}(x)}{\Gamma_{S_n}(\underline{s})\delta_{-\underline{s}}(y)}$$

The Laplace transform of $\gamma_{\underline{s},y}$:

$$\mathcal{L}(\gamma_{\underline{s},y})(z) = \frac{\delta_{-\underline{s}}(y+z)}{\delta_{-\underline{s}}(y)}$$

In the case of **one-dimensional shape parameter** $\underline{s} = s(1,\ldots,1)$, we have $\delta_{\underline{s}}(y) = (\det y)^s$ and

$$\mathcal{L}(\gamma_{s,y})(z) = (\det(y+z)\det(y^{-1}))^{-s} = \det(I+zy^{-1})^{-s}$$

Important direction of modern multivariate statistics:
Wishart laws and Riesz measures
on subcones Ω of S_n.

Cones of matrices with obligatory zeros and dual cones

WHY CONES WITH OBLIGATORY ZEROS APPEAR IN STATISTICS:

$X = (X_1, X_2, \ldots, X_n)^t$ a Gaussian vector $N(\mathbf{m}, \mathbf{\Sigma})$.

Some entries of the vector X are supposed to be conditionally independent knowing others

Conditional independence in a.c. case

$X = (X_1, X_2, X_3)$: Random vector
$f_{X_1,X_2,X_3}(x_1, x_2, x_3)$: density function

X_1 and X_3 are conditionally independent knowing X_2

$\Leftrightarrow f_{X_1,X_3|X_2=x_2} = f_{X_1|X_2=x_2} f_{X_3|X_2=x_2}$

$\Leftrightarrow f_{X_1,X_2,X_3}(x_1, x_2, x_3) = F(x_1, x_2) G(x_2, x_3)$

$X \sim N(0, \Sigma), \quad \Sigma \in S_3^+$

$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = (\det 2\pi\Sigma)^{-1/2} \exp(-{}^t x \Sigma^{-1} x/2)$

Put $\sigma := \Sigma^{-1}$. Mixing x_1 and x_3 can be avoided only when $\sigma_{13} = 0$:

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3)$$
$$= (2\pi)^{-3/2}(\det \sigma)^{1/2} \exp\Big(-(\sigma_{11}x_1^2 + 2\sigma_{12}x_1x_2 + \sigma_{22}x_2^2)/2\Big)$$
$$\times \exp\Big(-(2\sigma_{23}x_2x_3 + \sigma_{33}x_3^2)/2\Big)$$

Therefore,

$(X_1 \perp X_3)|X_2 \ \Leftrightarrow \sigma_{13} = 0$

The matrix $\sigma = \Sigma^{-1}$ has obligatory zeros $\sigma_{13} = \sigma_{31} = 0$

The position of zeros in Σ^{-1} is encoded by a graph

$G = (V, E)$: undirected graph
$V = \{1, \dots, r\}$: the set of vertices
$E \subset V \times V$: the set of edges
$i \sim j \Leftrightarrow (i, j) \in E$

$Z_G := \left\{ x \in \mathrm{Sym}(r, \mathbb{R}) \,|\, x_{ij} = 0 \text{ if } i \neq j \text{ and } i \not\sim j \right\}$
$P_G := Z_G \cap S_r^+$ a sub-cone of S_r^+

$X \sim N(0, \Sigma), \qquad \Sigma^{-1} \in P_G$
$\Leftrightarrow$ X_i and X_j are conditionally independent knowing all other components if $i \neq j$ and $i \not\sim j$

Example 1 $(X_1 \perp X_3) | X_2$ corresponds to G: 1–2–3

Example 1. Graph $G = A_3$: 1–2–3

$$Z_G := \left\{ \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{12} & x_{22} & x_{23} \\ 0 & x_{23} & x_{33} \end{pmatrix} \,\middle|\, x_{ij} \in \mathbb{R} \right\}$$

$P_G := Z_G \cap S_3^+$

This cone is homogeneous

($GL(P_G)$ acts transitively on P_G)

$$Z_G^* := \left\{ \begin{pmatrix} \xi_{11} & \xi_{12} & * \\ \xi_{12} & \xi_{22} & \xi_{23} \\ * & \xi_{23} & \xi_{33} \end{pmatrix} \,\middle|\, x_{ij} \in \mathbb{R} \right\}$$

$$\begin{aligned} P_G^* = Q_G &:= \left\{ \xi \in Z_G^* \,|\, \mathrm{tr}\, x\xi > 0 \text{ for all } x \in \overline{\Omega_1} \setminus \{0\} \right\} \\ &= \left\{ \xi \in Z_G^* \,\middle|\, \begin{vmatrix} \xi_{11} & \xi_{12} \\ \xi_{12} & \xi_{22} \end{vmatrix} > 0,\ \begin{vmatrix} \xi_{22} & \xi_{23} \\ \xi_{23} & \xi_{33} \end{vmatrix} > 0,\ \xi_{33} > 0 \right\} \end{aligned}$$

Example 2. Graph $G = A_4$: 1–2–3–4

$$Z_G := \left\{ \begin{pmatrix} x_{11} & x_{21} & 0 & 0 \\ x_{21} & x_{22} & x_{32} & 0 \\ 0 & x_{32} & x_{33} & x_{43} \\ 0 & 0 & x_{43} & x_{44} \end{pmatrix} \,\middle|\, x_{11}, \ldots, x_{44} \in \mathbb{R} \right\}$$

$P_G := Z_G \cap S_4^+$

This cone is non-homogeneous

$$P_G^* = Q_G := \left\{ \xi \in Z_G^* \,\middle|\, \mathrm{tr}\, x\xi > 0 \text{ for all } x \in \overline{\Omega_1} \setminus \{0\} \right\}$$

$$= \left\{ \xi \in Z_G^* \,\middle|\, \begin{vmatrix} \xi_{11} & \xi_{12} \\ \xi_{12} & \xi_{22} \end{vmatrix} > 0,\ \begin{vmatrix} \xi_{22} & \xi_{23} \\ \xi_{23} & \xi_{33} \end{vmatrix} > 0,\ \begin{vmatrix} \xi_{33} & \xi_{34} \\ \xi_{34} & \xi_{44} \end{vmatrix} > 0,\ \xi_{44} > 0 \right\}$$

Theory of graphical models
started in 1976 by Lauritzen and Speed,
is for decomposable graphs G

G is decomposable
$\Leftrightarrow$ G has no cycle of length ≥ 4 as an induced subgraph
Example: A_4 =1–2–3–4 from Example 2

$\Omega_G \subset Z_G$ is homogeneous if and only if
G is decomposable and A_4-free (Letac-Massam, Ishi)

Wishart distributions for decomposable graphs

A seminal paper:

G. Letac and H. Massam,
Wishart distributions for decomposable graphs,
The Annals of Statistics, **35** (2007), 1278–1323.

Letac-Massam power functions on Q_{A_n}

$$H(\alpha, \beta, \eta) = \frac{\prod_{i=1}^{n-1} |\eta_{\{i:i+1\}}|^{\alpha_i}}{\prod_{i=2}^{n-1} \eta_{ii}^{\beta_i}}$$

This definition comes from the graph theory
(CLIQUES $\{i, i+1\}$, SEPARATORS $\{i\}$)

Our approach to Wishart theory for decomposable graphs:

Consider analogs of "future" and "past" power functions

$$\delta_{\underline{s}}(x) \quad \text{and} \quad \Delta_{\underline{s}}(x)$$

for all eliminating orders of vertices

There are many (but not all) orders of vertices $1, 2, \ldots, n$ that we should consider in order to have a harmonious theory of Riesz and Wishart distributions on the cones related to graphs.

These orders are called *eliminating orders of vertices*.

Let v^+ be the set of future(w.r. to the order) neigbours (w.r. to the graph) of v.

An eliminating order of the vertices of G is a permutation $\{v_1, \ldots, v_n\}$ of V such that for all v, the set v^+ is a complete graph

Example. For the graph $A_3 : 1 - 2 - 3$:
the orders $1 \prec 2 \prec 3$, $1 \prec 3 \prec 2$, $3 \prec 2 \prec 1$ and $3 \prec 1 \prec 2$ are eliminating orders
$2 \prec 1 \prec 3$ and $2 \prec 3 \prec 1$ are not eliminating.

Proposition. All eliminating orders on A_n are obtained by an **intertwining of two sequences**
$1 \prec 2 \prec 3 < \ldots \prec M - 1 \prec M$
$n \prec n - 1 \prec \ldots \prec M + 2 \prec M + 1 \prec M$
for an $M \in V$.

Power functions

Notations:

$v^- =$ **all** the predecessors of v w.r. to $\prec$

$v^+ =$ future **neighbours** of v.

We define power functions

$$\Delta_{\underline{s}}^{\prec}(y) := \prod_{v \in V} \left(\frac{\det y_{\{v\} \cup v^-}}{\det y_{v^-}} \right)^{s_v} \qquad (y \in P_G),$$

$$\delta_{\underline{s}}^{\prec}(\eta) := \prod_{v \in V} \left(\frac{\det \eta_{\{v\} \cup v^+}}{\det \eta_{v^+}} \right)^{s_v} \qquad (\eta \in Q_G)$$

where $\det y_{\emptyset} = 1 = \det \eta_{\emptyset}$.

In this research and lecture:

RECENT RESULTS ON RIESZ MEASURES AND WISHART DISTRIBUTIONS FOR GRAPHS

$A_n = 1 - 2 - \ldots - n$

From now on,

$$G = A_n = 1 - 2 - \ldots - n$$

Q_{A_n} and P_{A_n} are important non-homogeneous($n \geq 4$) cones appearing in the statistical theory of graphical models

They correspond to the practical model of nearest neighbour interactions:

in the Gaussian character $(X_1, X_2, \ldots, X_n)$, non-neighbours X_i, X_j, $|i-j| > 1$ are conditionally independent with respect to other variables.

Theorem 0. Let M be the maximal element with respect to an eliminating order $\prec$, $M = 1, 2, \ldots, n$.
Then for all $y \in P_G$,

$$\delta_{\underline{s}}^{\prec}(\pi_{Z_G^*}(y^{-1})) = \Delta_{-\underline{s}}^{\prec}(y) = \Delta_{-\underline{s}}^{(M)}(y)$$

Proof: Direct computation.

Corollary. The power functions $\delta_{\underline{s}}^{\prec}(\eta)$ and $\Delta_{-\underline{s}}^{\prec}(y)$ depend only on M, the maximal element of $\prec$.

Formulas for the power functions may be written as:

$$\begin{aligned}\Delta_{\underline{s}}^{(M)}(y) \quad &= y_{11}^{s_1-s_2}|y_{\{1:2\}}|^{s_2-s_3}\ldots|y_{\{1:M-1\}}|^{s_{M-1}-s_M} \\ &\times|y|^{s_M} \\ &\times|y_{\{M+1:n\}}|^{s_{M+1}-s_M}\ldots y_{nn}^{s_n-s_{n-1}}\end{aligned}$$

For $2 \le M \le n-1$,

$$\delta_{\underline{s}}^{(M)}(\eta) \quad = \frac{\prod_{i=1}^{M-1}|\eta_{\{i:i+1\}}|^{s_i}\prod_{i=M+1}^{n}|\eta_{\{i-1:i\}}|^{s_i}}{\prod_{i=2}^{M-1}\eta_{ii}^{s_{i-1}}\cdot\eta_{MM}^{s_{M-1}-s_M+s_{M+1}}\cdot\prod_{i=M+1}^{n-1}\eta_{ii}^{s_{i+1}}}$$

= a Letac-Massam power function H

$\delta_{\underline{s}}^{(1)}, \delta_{\underline{s}}^{(n)}$ are not covered by Letac-Massam approach.

For $n \geq 2$ define $\varphi_n : Q_{A_n} \to \mathbb{R}_+$ by

$$\varphi_n(\eta) = \prod_{i=1}^{n-1} |\eta_{\{i,i+1\}}|^{-3/2} \prod_{i \neq 1,n} \eta_{ii}$$

For $n = 1$ set

$$\varphi_1(\eta) = \eta^{-1}.$$

We will see that φ_n is the characteristic function of the cone Q_{A_n}.

Laplace transform of power functions

Theorem 1. For all $n \geq 1$, $1 \leq M \leq n$ and $y \in P_{A_n}$,

$$\int_{Q_{A_n}} e^{-\mathrm{tr}(y\eta)} \delta_{\underline{s}}^{(M)}(\eta) \varphi_n(\eta) d\eta = \pi^{(n-1)/2} \Gamma_{Q_{A_n}}(\underline{s}) \Delta_{-\underline{s}}^{(M)}(y)$$

where $\Gamma_{Q_{A_n}}(\underline{s}) = \left\{ \prod_{i \neq M} \Gamma(s_i - \frac{1}{2}) \right\} \Gamma(s_M)$.

The integral converges if and only if $s_i > \frac{1}{2}$, for all $i \neq M$ and $s_M > 0$.

Theorem 2. For all $n \geq 1$, for all $1 \leq M \leq n$ and for all $\eta \in Q_{A_n}$,

$$\int_{P_{A_n}} e^{-\mathrm{tr}(y\eta)} \Delta_{\underline{s}}^{(M)}(y)dy = \pi^{(n-1)/2} \Gamma_{P_{A_n}}(\underline{s}) \delta_{-\underline{s}}^{(M)}(\eta) \varphi_n(\eta).$$

where $\Gamma_{P_{A_n}}(\underline{s}) = \left\{ \prod_{i \neq M} \Gamma(s_i + \frac{3}{2}) \right\} \Gamma(s_M + 1)$.

The integral converges if and only if $s_i > -\frac{3}{2}$, for all $i \neq M$ and $s_M > -1$.

Corrolary 3.

$$\left(\frac{4}{\pi^2}\right)^{\frac{n-1}{2}} \int_{P_{A_n}} e^{-\mathrm{tr}(y\eta)} dy = \varphi_n(\eta).$$

Thus, up to a factor, φ_n is the characteristic function of the cone Q_{A_n}.

Method: Recurrent constructions of the cones P_{A_n} and Q_{A_n} from the cones $P_{A_{n-1}}$ and $Q_{A_{n-1}}$.

(Two versions of A_{n-1}: $2 - \ldots - n$ and $1 - \ldots - (n-1)$)

Let $\Phi_n : \mathbb{R}^+ \times \mathbb{R} \times P_{A_{n-1}} \longrightarrow P_{A_n}$, $(a, b, z) \longmapsto y$

and $\tilde{\Phi}_n : \mathbb{R}^+ \times \mathbb{R} \times P_{A_{n-1}} \longrightarrow P_{A_n}$, $(a, b, z) \longmapsto \tilde{y}$

$$y = \begin{pmatrix} 1 & & & & \\ b & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \ldots\ldots\ldots & & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & z & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & & & & \\ b & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \ldots\ldots\ldots & & 0 & 1 \end{pmatrix}^T$$

$$\tilde{y} = \begin{pmatrix} 1 & & & & \\ 0 & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \ldots\ldots\ldots & & b & 1 \end{pmatrix}^T \begin{pmatrix} & & & 0 \\ & z & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & a \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 0 & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \ldots\ldots\ldots & & b & 1 \end{pmatrix}$$

The maps Φ_n and $\tilde{\Phi}_n$ are bijections.

Let $\Psi_n : \mathbb{R}^+ \times \mathbb{R} \times Q_{A_{n-1}} \longrightarrow Q_{A_n}$, $(\alpha, \beta, x) \longmapsto \eta$
and $\tilde{\Psi}_n : \mathbb{R}^+ \times \mathbb{R} \times Q_{A_{n-1}} \longrightarrow Q_{A_n}$, $(\alpha, \beta, x) \longmapsto \tilde{\eta}$

$$\eta = \pi\left(\begin{pmatrix} 1 & & & & \\ \beta & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \dots\dots\dots & & 0 & 1 \end{pmatrix}^T \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & x & \\ 0 & & & \end{pmatrix} \begin{pmatrix} 1 & & & & \\ \beta & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \dots\dots\dots & & 0 & 1 \end{pmatrix}\right)$$

$$\tilde{\eta} = \pi\left(\begin{pmatrix} 1 & & & & \\ 0 & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \dots\dots\dots & & \beta & 1 \end{pmatrix} \begin{pmatrix} & & & 0 \\ & x & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ 0 & & & & \\ \vdots & & & \ddots & \\ 0 & \dots\dots\dots & & \beta & 1 \end{pmatrix}^T\right)$$

The maps Ψ_n and $\tilde{\Psi}_n$ are bijections.

for all $M = 2, \dots, n$,

$$\Delta_{\underline{s}}^{(M)}(y) = a^{s_1} \Delta_{(s_2,\dots,s_n)}^{(M)}(z);$$

$$\delta_{\underline{s}}^{(M)}(\eta) = \alpha^{s_1} \delta_{(s_2,\dots,s_n)}^{(M)}(x).$$

For $M = 1, \dots, n-1$ we use $\tilde{y} = \tilde{\Phi}_n(a, b, z)$ and $\tilde{\eta} = \tilde{\Psi}_n(\alpha, \beta, x)$:

$$\Delta_{\underline{s}}^{(1)}(\tilde{y}) = a^{s_n} \Delta_{(s_1,\dots,s_{n-1})}^{(1)}(z);$$

$$\delta_{\underline{s}}^{(1)}(\tilde{\eta}) = \alpha^{s_n} \delta_{(s_1,\dots,s_{n-1})}^{(1)}(x).$$

Jacobians: $J(\Phi_n)(a, b, z) = J(\tilde{\Phi}_n)(a, b, z) = a$, $J(\Psi_n)(\alpha, \beta, x) = x_{22}$, $J(\tilde{\Psi}_n)(\alpha, \beta, x) = x_{n-1,n-1}$.

Proof of Theorem 1, $M > 1$: We proceed by induction

For $n = 1$,

$$\int_0^\infty e^{-y\eta}\delta_s^{(1)}(\eta)\varphi_{A_1}(\eta)d\eta = \int_0^\infty e^{-y\eta}\eta^{s-1}d\eta = \Gamma(s)y^{-s}.$$

Assume that the assertion holds for some number of vertices $n-1$.
Let $y = \Phi_n(a,b,z)$ and let us make the change of variable $\eta = \Psi_n(\alpha,\beta,x)$.
The induction hypothesis gives

$$\int_{Q_{A_{n-1}}} e^{-\operatorname{tr}(zx)} \delta^{(M)}_{(s_2,\ldots,s_n)}(x) \varphi_{A_{n-1}}(x) dx =$$

$$\pi^{(n-2)/2} \Big\{ \prod_{i \neq 1, M} \Gamma(s_i - \frac{1}{2}) \Big\} \Gamma(s_M) \Delta^{(M)}_{-(s_2,\ldots,s_n)}(z),$$

if and only if $s_i > \frac{1}{2}$, for all $i \neq M$ and $s_M > 0$.

The change of variable $\eta = \Psi_n(\alpha, \beta, x)$ gives $d\eta = x_{22} d\alpha d\beta dx$. Thus, we have

$$\int_{Q_{A_n}} e^{-\operatorname{tr}(y\eta)} \delta_{\underline{s}}^{(M)}(\eta) \varphi_{A_n}(\eta) d\eta$$

$$= \int_0^\infty \int_{-\infty}^\infty \int_{Q_{A_{n-1}}} e^{-(a\alpha + a x_{22}(b+\beta)^2 + \operatorname{tr}(zx))} \times$$

$$\times \quad \alpha^{s_1} \delta_{(s_2,\dots,s_n)}^{(M)}(x)\, x_{22}^{-1/2} \alpha^{-3/2} \varphi_{A_{n-1}}(x)\, x_{22} d\alpha d\beta dx$$

$$= \int_0^\infty \int_{-\infty}^\infty \int_{Q_{A_{n-1}}} e^{-(a\alpha + a x_{22}(b+\beta)^2 + \operatorname{tr}(zx))} \times$$

$$\times \quad \alpha^{s_1 - 3/2} \delta_{(s_2,\dots,s_n)}^{(M)}(x) \varphi_{A_{n-1}}(x) x_{22}^{1/2} d\alpha d\beta dx,$$

Now, use the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax_{22}(b+\beta)^2} d\beta = \pi^{1/2} a^{-1/2} x_{22}^{-1/2}$$

and the gamma integral

$$\int_{0}^{\infty} e^{-a\alpha} \alpha^{s_1 - 3/2} d\alpha = a^{-s_1 + 1/2} \Gamma(s_1 - \frac{1}{2}),$$

that is finite if and only if $s_1 > \frac{1}{2}$, we get

$$\int_{Q_{A_n}} e^{-\operatorname{tr}(y\eta)} \delta_{\underline{s}}^{(M)}(\eta) \varphi_{A_n}(\eta) d\eta$$

$$= \pi^{\frac{1}{2}} a^{-s_1} \Gamma(s_1 - \frac{1}{2}) \int_{Q_{A_{n-1}}} e^{-\operatorname{tr}(zx)} \delta_{(s_2,\ldots,s_n)}^{(M)}(x) \varphi_{A_{n-1}}(x) dx$$

$$= \pi^{\frac{1}{2}} a^{-s_1} \Gamma(s_1 - \frac{1}{2}) \pi^{\frac{n-2}{2}} \Big\{ \prod_{i \neq 1, M} \Gamma(s_i - \frac{1}{2}) \Big\} \Gamma(s_M) \Delta_{-(s_2,\ldots,s_n)}^{(M)}(z)$$

LETAC-MASSAM CONJECTURE

This conjecture was formulated in

G. Letac and H. Massam,
Wishart distributions for decomposable graphs,
The Annals of Statistics, **35** (2007), 1278–1323.

Recall Letac-Massam power functions on Q_{A_n}

$$H(\alpha, \beta, \eta) = \frac{\prod_{i=1}^{n-1} |\eta_{\{i:i+1\}}|^{\alpha_i}}{\prod_{i=2}^{n-1} \eta_{ii}^{\beta_i}}$$

The Laplace transform formula $\forall y \in P_{A_n}$

$$\int_{Q_{A_n}} e^{-\mathrm{tr}(y\eta)} H(\alpha,\beta,\eta)\varphi_{Q_{A_n}}(\eta)d\eta = C_{\alpha,\beta} H(\alpha,\beta,\pi^{-1}(y)),$$

will be referred to as the Letac-Massam (LM) formula on Q_{A_n}.

There are $2n-3$ parameters α, β in $H(\alpha, \beta, \cdot)$.
By [L-M], the LM formula holds for "'well chosen"' α, β, i.e. α, β veryfing Letac-Massam conditions:

(C) $\alpha_{j,j+1} = \beta_{j+1}$ *if* $1 \leq j \leq M-2$,
$\alpha_{j,j+1} = \beta_j$ *if* $M+1 \leq j \leq n-1$

(I) $\alpha_{j,j+1} > \frac{1}{2}$ for all $j = 1, \ldots, n-1$,
$\alpha_{M-1,M} + \alpha_{M,M+1} - \beta_M > 0$
for some $M = 2, \ldots, n-1$.

Remarks. (C) limits the number of "'free"' parameters α, β to n.

There are n parameters s_i indexing the power function $\delta_{\underline{s}}^{(M)}(\eta)$.

$H(\alpha, \beta, \eta) = \delta_{\underline{s}}^{(M)}(\eta)$ if and only if (C) holds true.

Recall

Theorem 1. For all $n \geq 1$, $1 \leq M \leq n$ and $y \in P_{A_n}$,

$$\int_{Q_{A_n}} e^{-\text{tr}(y\eta)} \delta_{\underline{s}}^{(M)}(\eta)\varphi_n(\eta)d\eta = \pi^{(n-1)/2}\Gamma_{Q_{A_n}}(\underline{s})\Delta_{-\underline{s}}^{(M)}(y)$$

where $\Gamma_{Q_{A_n}}(\underline{s}) = \left\{\prod_{i \neq M} \Gamma(s_i - \frac{1}{2})\right\}\Gamma(s_M)$.

The integral converges if and only if $s_i > \frac{1}{2}$, for all $i \neq M$ and $s_M > 0$.

Define $r_i = \alpha_i - \beta_{i+1}$, for all $1 \le i \le n-3$ and $p_i = \alpha_i - \beta_i$, for all $3 \le i \le n-1$. We have

$$H(\alpha, \beta, \eta) = \delta_{\underline{s}}^{(M)}(\eta) \prod_{i=2}^{M-1} \eta_{ii}^{r_{i-1}} \prod_{i=M+1}^{n-1} \eta_{ii}^{p_i},$$

where $s_i = \alpha_i$, for all $1 \le i \le M-1$; $s_i = \alpha_{i-1}$, for all $M+1 \le i \le n$ and $\beta_M = s_{M-1} - s_M + s_{M+1}$.

We have proved

L-M CONJECTURE Letac-Massam formula on Q_{A_n} holds if and only if conditions (C) and (I) are satisfied

Recall that (C) is equivalent to

$$H(\alpha, \beta, \eta) = \delta_{\underline{s}}^{(M)}(\eta)$$

for some $M = 2, \ldots, n-1$

(I) is equivalent to "$\delta_{\underline{s}}^{(M)}$ admits Laplace transform"

Thus the functions $\delta_{\underline{s}}^{(M)}$ are more natural as power functions on Q_G than $H(\alpha, \beta, \eta)$.

OUTLINE OF THE PROOF OF the L-M CONJECTURE on Q_{A_n}

Letac-Massam Conjecture for power functions $\delta_{\underline{s}}^{(M)}$ and $\Delta_{\underline{s}}^{(M)}$

Let $\varphi(y) = \pi(y^{-1})$.

The Letac-Massam formula is equivalent,
for each $2 \le M \le n-1$, to

$$\int_{Q_{A_n}} e^{-\operatorname{tr}(y\eta)} \delta_{\underline{s}}^{(M)}(\eta) \prod_{i=2}^{M-1} \eta_{ii}^{r_i-1} \prod_{i=M+1}^{n-1} \eta_{ii}^{p_i} \varphi_{Q_{A_n}}(\eta) d\eta$$

$$= C_{\alpha,\beta} \Delta_{-\underline{s}}^{(M)}(y) \prod_{i=2}^{M-1} \varphi(y)_{ii}^{r_i-1} \prod_{i=M+1}^{n-1} \varphi(y)_{ii}^{p_i}.$$

The Letac-Massam conditions (C) are equivalent to the following $n-2$ alternative conditions:

$$\begin{array}{lllll} p_3 = & p_4 = & \ldots = & p_{n-1} = 0 & or \\ r_1 = & p_4 = & \ldots = & p_{n-1} = 0 & or \\ \vdots & \vdots & \vdots & \vdots & or \\ r_1 = & \ldots = & r_{n-4} = & p_{n-1} = 0 & or \\ r_1 = & \ldots = & r_{n-4} = & r_{n-3} = 0. & \end{array} \quad (1)$$

We express, for each M, the constant $C_{\alpha,\beta}$ as a function of $M, \underline{s} = (s_i), (r_i)$ and (p_i).

Lemma 4. If the LM formula holds for all $y \in P_{A_n}$ then we have

$$C_{\alpha,\beta} = \pi^{(n-1)/2} \times$$
$$\left\{ \prod_{i \neq M} \Gamma(s_i - \frac{1}{2}) \right\} \Gamma(s_M) \prod_{2 \leq i < M} \frac{\Gamma(s_i + r_{i-1})}{\Gamma(s_i)} \prod_{M < i \leq n-1} \frac{\Gamma(s_i + p_i)}{\Gamma(s_i)}.$$

If y is diagonal, then LM formula holds if and only if $s_i > \frac{1}{2}$ for $i \neq M$, $s_m > 0$, $s_i + r_{i-1} > 0$ for $2 \leq i < M$ and $s_i + p_i > 0$ for $M < i \leq n-1$.

Proof We take y diagonal. The proof is a by-product of the main induction proof.

We prove the Letac-Massam conjecture by induction on n. The proof of the initiation part ($n = 4$) and the heredity part ($n \geq 5$) are the same, so they are given together.

Step 1 (descent in Letac-Massam formula, from Q_{A_n} to $Q_{A_{n-1}}$).
Let $n \geq 4$, $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ and $\beta = (\beta_2, \ldots, \beta_{n-1})$. Suppose that the Letac-Massam formula holds for $H_n(\alpha, \beta, \cdot)$ on Q_{A_n}. Then the Letac-Massam formula holds on $Q_{A_{n-1}}$ for:
(i) the function $H_{n-1}((\alpha_1, \ldots, \alpha_{n-2}), (\beta_2, \ldots, \beta_{n-2}), \cdot)$ and the graph $1 - \ldots - (n-1)$
(ii) the function $H_{n-1}((\alpha_2, \ldots, \alpha_{n-1}), (\beta_3, \ldots, \beta_{n-1}), \cdot)$ and the graph $2 - \ldots - n$.

Proof of Step 1. We choose $2 \leq M \leq n-2$. For all $y \in P_{A_n}$, let, successively, $y = \tilde{\Phi}_n(a,b,z)$ and $z = \Phi_{n-1}(a,b,Z)$. One easily checks that $\varphi(y)_{ii} = \varphi(z)_{ii} = \varphi(Z)_{ii}$. Integration on Q_{A_n} with two successive changes of variables $\eta = \tilde{\Psi}_n(\alpha,\beta,x)$ and then $x = \Psi_{n-1}(\alpha,\beta,X)$ gives

$$\int_{Q_{A_{n-2}}} e^{-\operatorname{tr}(ZX)} \delta^{(M)}_{(s_2,\dots,s_{n-1})}(X) \prod_{i=2}^{M-1} X_{ii}^{r_i-1} \prod_{i=M+1}^{n-1} X_{ii}^{p_i} \varphi_{Q_{A_{n-2}}}(X)dX \tag{2}$$

$$= C^{(n-2)}_{\alpha,\beta} \Delta^{(M)}_{-(s_2,\dots,s_{n-1})}(Z) \prod_{i=2}^{M-1} \varphi(Z)_{ii}^{r_i-1} \prod_{i=M+1}^{n-1} \varphi(Z)_{ii}^{p_i}.$$

where $C^{(n-2)}_{\alpha,\beta} = \frac{C_{\alpha,\beta}}{\pi\Gamma(s_1-\frac{1}{2})\Gamma(s_n-\frac{1}{2})}$.

Now, we apply one more change of variable $X = \tilde{\Psi}_{n-2}(\alpha, \beta, U)$ in formula (2) and we set $Z = \tilde{\Phi}_{n-2}(a, 0, T)$. Let $F(\alpha, \beta, U)$ be the integrated function. We first compute $J = \int_{-\infty}^{\infty} \int_0^{\infty} F d\alpha d\beta = 2 \int_0^{\infty} \int_0^{\infty} F d\alpha d\beta$. Using the change of variables $u = a\alpha, t = aU_{n-2,n-2}\beta^2$ we get

$$J = 2a^{-p_{n-1}} \times$$
$$\int_0^{\infty} \int_0^{\infty} e^{-\left(a\alpha + aU_{n-2,n-2}\beta^2\right)} \alpha^{s_{n-1}-\frac{3}{2}} (a\alpha + a\beta^2 U_{n-2,n-2})^{p_{n-1}} d\alpha d\beta =$$
$$a^{-(s_{n-1}+p_{n-1})} U_{n-2,n-2}^{-1/2} \int_0^{\infty} \int_0^{\infty} e^{-(u+t)} u^{s_{n-1}-\frac{3}{2}} t^{-\frac{1}{2}} (u+t)^{p_{n-1}} du dt$$

Now, using the change of variables $u = u, v = u + t$, we get (with a change of variable $x = u/v$)

$$J = a^{-(s_{n-1}+p_{n-1})} U_{n-2,n-2}^{-1/2} \int_0^{\infty} \left(\int_0^{v} u^{s_{n-1}-\frac{3}{2}} (v-u)^{-\frac{1}{2}} du \right) e^{-v} v^{p_{n-1}} dv$$
$$= a^{-(s_{n-1}+p_{n-1})} U_{n-2,n-2}^{-1/2} B(s_{n-1} - \frac{1}{2}, \frac{1}{2}) \Gamma(s_{n-1} + p_{n-1}) \tag{3}$$

We get

$$\int_{Q_{A_{n-3}}} e^{-\operatorname{tr}(TU)} \delta^{(M)}_{(s_2,\dots,s_{n-2})}(U) \prod_{i=2}^{M-1} U_{ii}^{r_i-1} \prod_{i=M+1}^{n-2} U_{ii}^{p_i} \varphi_{Q_{A_{n-3}}}(U)dU \tag{4}$$

$$= \quad C^{(n-3)}_{\alpha,\beta} \Delta^{(M)}_{-(s_2,\dots,s_{n-2})}(T) \prod_{i=2}^{M-1} \varphi(T)_{ii}^{r_i-1} \prod_{i=M+1}^{n-2} \varphi(T)_{ii}^{p_i},$$

where

$$C^{(n-3)}_{\alpha,\beta} = \frac{C_{\alpha,\beta}}{\pi^{\frac{3}{2}}\Gamma(s_1-\frac{1}{2})\Gamma(s_n-\frac{1}{2})\Gamma(s_{n-1}-\frac{1}{2})} \times \frac{\Gamma(s_{n-1})}{\Gamma(p_{n-1}+s_{n-1})}. \tag{5}$$

By the same argument as to obtain formula (2), we observe that the Letac-Massam formula pour la fonction $H_{n-1}((\alpha_1,\dots,\alpha_{n-2}),(\beta_2,\dots,\beta_{n-2}),\cdot)$ on $Q_{A_{n-1}}$ and the graph $1-2-\dots-(n-1)$ is equivalent to formula (4).

By a mirror argument, with the change of variables $X = \Psi_{n-2}(\alpha, \beta, U)$ in (2), we get the Letac-Massam formula for $H_{n-1}((\alpha_2, \ldots, \alpha_{n-1}), (\beta_3, \ldots, \beta_{n-1}), \cdot)$ and the graph $2 - \ldots - n$.

Proof of Lemma 4. For y diagonal, formula (5) leads by induction to formula from Lemma 4, observing that the last equation we get is

$$a^{-s_M} \int_0^\infty e^{-ax} x^{s_M} \frac{dx}{x} = C^{(1)}_{\alpha,\beta} a^{-s_M}$$

so that $C^{(1)}_{\alpha,\beta} = \Gamma(s_M)$.

Step 2 (induction step). *The Letac-Massam conjecture on $Q_{A_{n-1}}$ implies the Letac-Massam conjecture on Q_{A_n}.*

Proof. Let $n \geq 4$. Suppose that the Letac-Massam formula holds for some α and β and suppose that the Letac-Massam conjecture is true on $Q_{A_{n-1}}$.

For $n \geq 5$, we use Step 1 and the induction hypothesis. Thus one of the following $n-3$ conditions has to be satisfied: for an $M \in \{2, \ldots, n-2\}$

$$r_1 = \ldots = r_{M-2} = p_{M+1} = \ldots = p_{n-2} = 0,$$

and, simultaneously, one of the following $n-3$ "shifted" conditions has to be satisfied: for an $M \in \{2, \ldots, n-2\}$

$$r_2 = \ldots = r_{M-1} = p_{M+2} = \ldots = p_{n-1} = 0.$$

This implies that either conditions (C) are satisfied or

$$p_3 = \ldots = p_{n-2} = 0; r_2 = \ldots = r_{n-3} = 0. \qquad (6)$$

Let us assume this exceptional case. The equality $r_{M-1} = 0$ implies $s_M = s_{M+1}$ and $p_M = 0$ implies $s_M = s_{M-1}$. Also, from $p_j = r_{j-1}$ for all $3 \le j \le n-2$, we get $s_2 = \ldots = s_{M-1}$ and $s_{M+1} = \ldots = s_{n-1}$. Thus, $s_2 = \ldots = s_{n-1} = s$. In the case (6), using the formula for $\varphi(Z)_{ii}$, formula (2) reduces to

$$\int_{Q_{A_{n-2}}} e^{-\operatorname{tr}(ZX)} \delta^{(M)}_{(s,\ldots,s)}(X) X_{22}^{r_1} X_{n-1,n-1}^{p_{n-1}} \varphi_{Q_{A_{n-2}}}(X) dX$$

$$= C^{(n-2)}_{\alpha,\beta} |Z|^{-s} \left(\frac{|Z_{\{3:n-1\}}|}{|Z|} \right)^{r_1} \left(\frac{|Z_{\{2:n-2\}}|}{|Z|} \right)^{p_{n-1}} . \qquad (7)$$

A TRICK: take SECOND DERIVATIVE with respect to $Z_{n-2,n-1}$ and restrain to $Z_{n-2,n-1} = 0$

The derivatives of all orders of the integral (7) can be computed under the integral sign. We obtain

$$\int_{Q_{A_{n-2}}} e^{-\operatorname{tr}(ZX)} \delta^{(M)}_{(s,\dots,s)}(X) X_{22}^{r_1} X_{n-1,\,n-1}^{p_{n-1}} X_{n-2,n-1}^{2} \varphi_{Q_{A_{n-2}}}(X)\, dX$$
$$= \frac{C^{(n-2)}_{\alpha,\beta}}{4} \frac{\partial^2}{\partial Z^2_{n-2,n-1}} \left(|Z|^{-s} \left(\frac{|Z_{\{3:n-1\}}|}{|Z|} \right)^{r_1} \left(\frac{|Z_{\{2:n-2\}}|}{|Z|} \right)^{p_{n-1}} \right). \quad (8)$$

LHS: Let us change the variables $X = \tilde{\Psi}_{n-2}(\alpha, \beta, U)$ and set $Z = \tilde{\Phi}_{n-2}(a, 0, T)$, i.e. $Z_{n-2,n-1} = 0$. Similarly as in the proof of (4) in Step 1, we find that the left hand side of (8) is

$$a^{-(s+p_{n-1}+1)}\Gamma(s+p_{n-1}+1)B\left(s-\tfrac{1}{2}, \tfrac{3}{2}\right) \times \int_{Q_{A_{n-3}}} e^{-\operatorname{tr}(TU)}\delta^{(M)}_{(s,\ldots,s)}(U)U_{22}^{r_1}U_{n-2,n-2}\varphi_{Q_{A_{n-3}}}(U)dU. \quad (9)$$

RHS is standard, using Leibniz formula. We get that for $Z_{n-2,n-1} = 0$, the right hand side of (8) is

$$\frac{C_{\alpha,\beta}^{(n-2)}}{2} a^{-(s+p_{n-1}+1)} |T|^{-(s+r_1+1)} |T_{\{3:n-2\}}|^{r_1-1} \times \left[(s + r_1 + p_{n-1}) |T_{\{3:n-2\}}| |T_{\{2:n-3\}}| - r_1 |T_{\{3:n-3\}}| |T| \right]. \tag{10}$$

Equating (10) and (9), we obtain

$$\int_{Q_{A_{n-3}}} e^{-\mathrm{tr}(TU)}\delta^{(M)}_{(s,\ldots,s)} U_{22}^{r_1} U_{n-2,n-2}\varphi_{Q_{A_{n-3}}}(U)dU = \frac{sd(s,r_1,T)}{s+p_{n-1}}\Big[(s+r_1+p_{n-1})|T_{\{3:n-2\}}||T_{\{2:n-3\}}| - r_1|T_{\{3:n-3\}}||T|\Big], \tag{11}$$

where $d(s,r_1,T) = C^{(n-3)}_{\alpha,\beta}|T|^{-(s+r_1+1)}|T_{\{3:n-2\}}|^{r_1-1}$.

Formula (11) is supposed to be true for our $p_{n-1} = \alpha_{n-1} - \beta_{n-1}$. It is surely true for $p_{n-1} = 0$, because the Letac-Massam conditions (1) are then satisfied. Equating (11) for these two values of p_{n-1}, and noting that by Lemma 4 the constant $C^{(n-3)}_{\alpha,\beta}$ does not depend on p_{n-1}, we get

$$\frac{s[(s+r_1+p_{n-1})|T_{\{3:n-2\}}||T_{\{2:n-3\}}| - r_1|T_{\{3:n-3\}}||T|]}{s+p_{n-1}} =$$
$$= (s+r_1)|T_{\{3:n-2\}}||T_{\{2:n-3\}}| - r_1|T_{\{3:n-3\}}||T|,$$

which is equivalent to

$$r_1 p_{n-1}\left(|T_{\{3:n-2\}}||T_{\{2:n-3\}}| - |T_{\{3:n-3\}}||T|\right) = 0,$$

where for $n=5$ we set $|T_{\{3:n-3\}}| = 1$.

We observe that $|T_{\{3:n-2\}}||T_{\{2:n-3\}}| - |T_{\{3:n-3\}}||T| \neq 0$, for example for T such that $T_{ii} = 2$ for all $2 \leq i \leq n-2$, $T_{i,i+1} = T_{i+1,i} = 1$ for $2 \leq i \leq n-3$ and $T_{ij} = 0$ for all other i, j (in this case, this expression equals 1).

Thus, for $n \geq 5$, in the exceptional case (6), we also have $r_1 = 0$ or $p_{n-1} = 0$.

In both cases we fall in the Letac-Massam conditions (C) and the proof of the induction step is finished.

For $n = 4$, we get formula (2) for $M = 2$, the computations are simpler (no use of Leibniz formula is needed), and no condition $s_2 = s_3 = s$ appears. The analogue of formula (11) is

$$\Gamma(s_3 + p_3 + 1) B(s_3 - \frac{1}{2}, \frac{3}{2}) \int_0^\infty e^{-tu} u^{s_2} u \frac{1}{u} du =$$
$$\frac{C^{(2)}_{\alpha,\beta}}{2} (s_2 + p_3) t^{-(s_2+1)}, \quad t > 0.$$

After substitution of the constant
$C^{(2)}_{\alpha,\beta} = \pi^{\frac{1}{2}} \Gamma(s_2) \Gamma(s_3 - \frac{1}{2}) \frac{\Gamma(s_3+p_3)}{\Gamma(s_3)}$
one gets

$$(s_3 + p_3) s_2 = s_3 (s_2 + p_3)$$

equivalent to $r_1 p_3 = 0$, so $r_1 = 0$ or $p_3 = 0$.

We get the Letac-Massam conditions for Q_{A_4}.