Proof: (i) $\mathrm{FDR}(R^{BH}, P) = \sum_{j=1}^{m} (1-\theta_j)\, \mathbb{E}\left[\frac{\mathbb{1}\{p_j \leq \alpha(\hat{\ell}\vee 1)/m\}}{\hat{\ell}\vee 1}\right]$

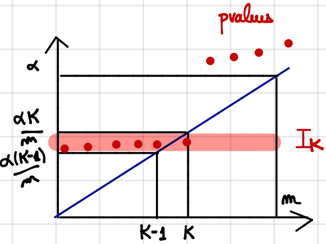
Use that for any $\ell \in \{1,\dots,m\}$, $\frac{1}{\ell} = \sum_{k\geq \ell}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k\geq\ell} \frac{1}{k(k+1)}$

Therefore $\mathbb{E}\left[\frac{\mathbb{1}\{p_j \leq \alpha(\hat{\ell}\vee 1)/m\}}{\hat{\ell}\vee 1}\right] \overset{\text{Fubini}}{=} \mathbb{E}\left[\sum_{k=1}^{m} \frac{1}{k(k+1)} \mathbb{1}\{p_j \leq \alpha(\hat{\ell}\vee 1)/m\}\, \mathbb{1}\{k \geq \hat{\ell}\vee 1\}\right]$

$$\leq \sum_{k\geq 1} \frac{1}{k(k+1)} \mathbb{P}\left(p_j \leq \frac{\alpha(k\wedge m)}{m}, \cancel{k \geq \hat{\ell}\vee 1}\right) \leq \frac{\alpha}{m} \sum_{k\geq 1} \frac{k\wedge m}{k(k+1)}$$ because $\theta_j = 0$

$$\sum_{k\geq 1} \frac{k\wedge m}{k(k+1)} = \sum_{k=1}^{m} \frac{1}{k+1} + \sum_{k\geq m} m \frac{1}{k(k+1)} = 1 + \frac{1}{2} + \dots + \frac{1}{m}.$$

(ii) <u>assume $\alpha\gamma_m \leq 1$</u>. We build a special distribution on $[0,1]^m$ in the following way



Let $I_k = \left(\frac{\alpha(k-1)}{m}, \frac{\alpha k}{m}\right]$, $1 \leq k \leq m$ and $I_0 = (\alpha, 1]$

* K is drawn on $\{0,1,\dots,m\}$ with $\begin{cases} P(K=k) = \alpha/k, & 1\leq k\leq m \\ P(K=0) = 1-\alpha\gamma_m \end{cases}$
* conditionnally on $\{K=k\}$, pick S_k uniformly in the subsets of $\{1,\dots,m\}$ of cardinal k
  - pick U_j, $j\in S_k$, iid uniform in I_k
  - pick U_j, $j \notin S_k$ iid uniform in I_0

Then each U_j has a uniform distribution:

$$\forall j \in \{1,\dots,m\}, \quad \forall k \in \{1,\dots,m\}, \quad \mathbb{P}(U_j \in I_k) = P(U_j \in I_k, K=k) = P(j\in S_k, K=k)$$
$$= P(j \in S_k \mid K=k) \times P(K=k)$$
$$= \frac{k}{m} \times \alpha/k = \frac{\alpha}{m}$$

and, clearly, U_j uniform on I_k cond. on "$U_j \in I_k$"

Now, by assumption, $\exists P_0 \in \mathcal{P}$ such that $\theta(P_0) = 0$ (full null) and

$$\underset{X\sim P_0}{(p_j(X), 1\leq j\leq m) \sim (U_j, 1 \leq j \leq m)}$$

We merely check that the number of rejections of BH procedure satisfies $\hat{\ell} \sim K$

Therefore $\mathrm{FDR}(R^{BH}, P_0) = \mathbb{P}(\hat{\ell} \geq 1) = P(K \geq 1) = \alpha\gamma_m$

If $\alpha \gamma_m > 1$ ie $\alpha > \gamma_m^{-1} := \alpha^*$ choose P_0 as before with α^* instead of α (so that $\alpha^* \gamma_m = 1$)

$$FDR(R^{BH(\alpha)}, P_0) = P(|R^{BH(\alpha)}| \geq 1)$$
$$\geq P(|R^{BH(\alpha^*)}| \geq 1) = FDR(R^{BH(\alpha^*)}, P_0) \underset{\text{from above}}{=} 1 \qquad \square$$

Consequence: $R^{BH(\frac{\alpha}{\gamma_m})}$ always controls the FDR at level α. called the BY procedure

Exercise generalization of BY procedure [Blanchard and R. (2008)]

Consider $\beta(\ell) = \sum_{k=1}^{\ell} k \nu_k$ with $\{\nu_k, 1 \leq k \leq m\}$ some proba measure

Show that the step-up stopping rule $\tilde{\ell} = \max\{\ell \in \{0, .., m\} : p_{(\ell)} \leq \alpha \frac{\beta(\ell)}{m}\}$ always provides a control at level α.

But: known to be conservative procedure for a 'realistic' P

so users prefer BH

2) Dependence conditions ensuring that BH control the FDR

Theorem: consider p-values satisfying (*) and the BH procedure at level α with $\hat{\ell}$ rejections

Assume that $P \in \mathcal{P}$ is such that

$$\forall j \text{ with } \theta_j = 0, \forall \ell \in \{2, \dots, m\}, \mathbb{P}\left(\hat{\ell} \leq \ell - 1 \mid p_j \leq \frac{\alpha(\ell-1)}{m}\right) \leq \mathbb{P}\left(\hat{\ell} \leq \ell - 1 \mid p_j \leq \frac{\alpha \ell}{m}\right) \quad (\#)$$

Then $FDR(R^{BH}, P) \leq \frac{m_0(P)}{m} \alpha$

Proof: $FDR(R^{BH}, P)$

$$= \sum_{j=1}^{m} (1-\theta_j) \sum_{\ell=1}^{m} \frac{1}{\ell} P\left(\hat{\ell} = \ell, p_j \leq \frac{\alpha \ell}{m}\right)$$

$$= \sum_{j=1}^{m} (1-\theta_j) \sum_{\ell=1}^{m} \frac{1}{\ell} \left[P\left(\hat{\ell} \leq \ell, p_j \leq \frac{\alpha \ell}{m}\right) - P\left(\hat{\ell} \leq \ell - 1, p_j \leq \frac{\alpha \ell}{m}\right)\right]$$

$$= \sum_{j=1}^{m} (1-\theta_j) \sum_{\ell=1}^{m} \frac{1}{\ell} \left[P\left(\hat{\ell} \leq \ell \mid p_j \leq \frac{\alpha \ell}{m}\right) - \underbrace{P\left(\hat{\ell} \leq \ell - 1 \mid p_j \leq \frac{\alpha \ell}{m}\right)}_{\geq P\left(\hat{\ell} \leq \ell - 1 \mid p_j \leq \frac{\alpha(\ell-1)}{m}\right) \text{ by } (\#)}\right] \overbrace{P\left(p_j \leq \frac{\alpha \ell}{m}\right)}^{\leq \frac{\alpha \ell}{m} \text{ by } (*)}$$

Hence $\mathrm{FDR}(R^{BH}, P)$

$$\leq \frac{\alpha}{m} \sum_{j=1}^{m} (1-\theta_j) \sum_{\ell=1}^{m} \left[P\left(\hat{\ell} \leq \ell \mid p_j \leq \frac{\alpha \ell}{m}\right) - P\left(\hat{\ell} \leq \ell - 1 \mid p_j \leq \frac{\alpha(\ell-1)}{m}\right) \right]$$

telescopic sum!

$$= \frac{\alpha}{m} \sum_{j=1}^{m} (1-\theta_j)\, P(\hat{\ell} \leq m \mid p_j \leq \alpha) = \frac{\alpha\, m_0}{m}$$

$\square$

How to ensure (#) ?

Application 1: positive dependence [Benjamini and Yekutieli (2001)]

$D_\ell = \{ p \in [0,1]^m : \hat{\ell}(p) \leq \ell - 1 \}$ is a non decreasing set of $[0,1]^m$

i.e. $\forall p, p' \in [0,1]^m$, $p \in D_\ell$ and $p_j \leq p'_j \ \forall j$ implies $p' \in D_\ell$

(#) holds if

$$\boxed{\begin{array}{l} \text{for all } D \subset [0,1]^m \text{ non decreasing (measurable)} \\ \text{for all } j \in \mathcal{H}_0(P) \\ \qquad t \in [0,1] \longmapsto \mathbb{P}(p \in D \mid p_j \leq t) \text{ is } \nearrow \end{array}}$$

weak PRDS (positively regressively dependent on each one of a subset)

For instance:

* weak PRDS satisfied in the Gaussian one sided setting when $\Gamma_{ij} \geq 0$ for all i,j

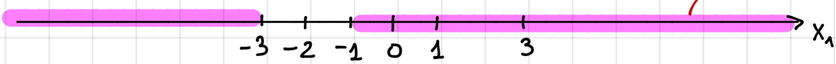
$\left[\text{use for all } j,\ \mathcal{D}\left((X_{j'})_{j' \neq j} \mid X_j = y\right) = \mathcal{N}_{m-1}\left((\mu_{j'})_{j' \neq j} + \Gamma_{\neq j, j}(y - \mu_j),\ \underbrace{\Gamma'}_{\text{indep of } y}\right)\right]$

* weak PRDS not necessarily satisfied in the Gaussian two-sided setting when $\Gamma_{ij} \geq 0$

for instance $m=2$, $\Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mu = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ then $X_2 = X_1 + 2$ a.s.

we have $\begin{cases} P(|X_2| > 1 \mid |X_1| \leq 1) = 1 \\ P(|X_2| > 1 \mid |X_1| \leq 3) < 1 \end{cases}$

set $|X_1 - 2| > 1$



Open problem: find $\sup\limits_{\substack{P \text{ Gaussian} \\ \text{one sided} \\ (\mu, \Gamma)}} \{ \mathrm{FDR}(R^{BH}, P) \}$ should be not so far from α

show $\sup\limits_{\substack{P \text{ Gaussian} \\ \text{two sided} \\ (\mu, \Gamma)}} \{ \mathrm{FDR}(R^{BH}, P) \} \leq \alpha$ satisfied in cases that are usually 'worst cases'

Application 2 Martingale dependence [Storey et al (2004)] [Heesen Janssen (2015)]

$D_\ell = \{\hat{\ell} \le \ell-1\}$ is measurable with respect to the filtration (with time running backwards)

$$\mathcal{F}_\ell = \sigma\left(\mathbb{1}\{p_j \le \tfrac{\alpha \ell'}{m}\},\ \ell \le \ell' \le m,\ 1 \le j \le m\right),\quad 1 \le \ell \le m$$

Indeed $\hat{\ell} \le \ell-1 \iff \forall \ell' \ge \ell,\ p_{(\ell')} > \frac{\alpha \ell'}{m} \iff \forall \ell' \ge \ell,\ \sum_{j=1}^{m} \mathbb{1}\{p_j \le \frac{\alpha \ell'}{m}\} < \ell'$

Hence
$$P\left(\hat{\ell} \le \ell-1 \mid p_j \le \frac{\alpha(\ell-1)}{m}\right) = \mathbb{E}\left[\mathbb{1}\{\hat{\ell} \le \ell-1\} \frac{\mathbb{1}\{p_j \le \frac{\alpha(\ell-1)}{m}\}}{P(p_j \le \frac{\alpha(\ell-1)}{m})}\right]$$
$$= \mathbb{E}\left[\mathbb{1}\{\hat{\ell} \le \ell-1\}\ \mathbb{E}\left[\underbrace{\frac{\mathbb{1}\{p_j \le \frac{\alpha(\ell-1)}{m}\}}{P(p_j \le \frac{\alpha(\ell-1)}{m})}}_{=:\, M^{(j)}_{\ell-1}} \,\Big|\, \mathcal{F}_\ell\right]\right]$$

If $\boxed{\forall \ell \in \{2, \dots, m\},\ \forall j \in \mathcal{H}_0(P),\ \mathbb{E}[M^{(j)}_{\ell-1} \mid \mathcal{F}_\ell] \le M^{(j)}_\ell}$ martingale-type condition

Then $\le \mathbb{E}\left[\mathbb{1}\{\hat{\ell} \le \ell-1\} M^{(j)}_\ell\right] = P\left(\hat{\ell} \le \ell-1 \mid p_j \le \frac{\alpha \ell}{m}\right)$ and (#) holds

Exercise show that martingale condition is satisfied under independence and directly prove FDR control by using that $\hat{\ell}$ is a stopping time

A non positive dependence configuration covered by the martingale condition:

Choose $(p_j, j \in \mathcal{H}_0)$ iid uniform and for $j \notin \mathcal{H}_0$, any p_j in $[0, \min_{j \in \mathcal{H}_0} p_j]$

Then

$$P\left(p_j \le \frac{\alpha(\ell-1)}{m} \mid \mathcal{F}_\ell\right) = P\left(p_j \le \frac{\alpha(\ell-1)}{m} \mid \mathbb{1}\{p_{j'} \le \frac{\alpha \ell'}{m}\},\ \ell \le \ell' \le m,\ 1 \le j' \le m\right) \times \mathbb{1}\{p_j \le \frac{\alpha \ell}{m}\}$$

$$= P\left(p_j \le \frac{\alpha(\ell-1)}{m} \mid p_j \le \frac{\alpha \ell}{m},\ p_{j'} \le \frac{\alpha \ell'}{m} \text{ pour } j' \ne j \text{ et } j' \in \mathcal{H}_0\right) \times \mathbb{1}\{p_j \le \frac{\alpha \ell}{m}\}$$

car $p_j \geq p_{j'}$ si $j' \notin \mathcal{H}_0$

$$= P\left(p_j \le \frac{\alpha(\ell-1)}{m} \mid p_j \le \frac{\alpha \ell}{m}\right) \times \mathbb{1}\{p_j \le \frac{\alpha \ell}{m}\}$$

by independence inside $\mathcal{H}_0$ so same as under indep

③ Adaptation to dependence

[Lehmann Romano (2005), Romano Wolf (2007), Guo et al (2014)]

* FDR control does not really allow to incorporate the dependence (recall $\mathrm{FDR} = \mathbb{E}[\mathrm{FDP}]$)
* we should control the fluctuations of the FDP

New goal: find $t = \hat{t}$ such that $\forall P \in \mathcal{P}, \ \mathbb{P}(\mathrm{FDP}(\hat{t}, P) \leq \alpha) \geq 1 - \delta$

often referred to as 'FDP control'

Via a corrected step up procedure: find new critical values τ_ℓ, $1 \leq \ell \leq m$,
such that for $\hat{\ell} = \max\{\ell \in \{0, 1, \dots, m\} : p_{(\ell)} \leq \tau_\ell\}$
the threshold $t = \tau_{\hat{\ell}}$ controls the FDP

Default critical values $\quad \tau_\ell = \delta \dfrac{\lfloor \alpha \ell \rfloor + 1}{m}, \quad 1 \leq \ell \leq m$ $\quad$ Lehmann Romano critical values

Controls the FDP under weak PRDS assumption [Guo et al (2014)]

> **Proposition**: consider pvalues satisfying (*) and $P \in \mathcal{P}$ satisfying the PRDS assumption
> consider the step-up procedure with Lehmann Romano critical values
> then $\mathbb{P}(\mathrm{FDP}(\tau_{\hat{\ell}}, P) \leq \alpha) \geq 1 - \delta$

Proof:

$$\{\mathrm{FDP}(\tau_{\hat{\ell}}) > \alpha\} = \Big\{ \sum_{j=1}^{m} (1-\theta_j)\, \mathbb{1}\{p_j \leq \tau_{\hat{\ell}}\} \geq \lfloor \alpha \hat{\ell} \rfloor + 1 \Big\}$$

$\to$ implies $m_0 \geq \lfloor \alpha \hat{\ell} \rfloor + 1$

$$\subset \Big\{ \exists \underset{\lfloor \alpha \hat{\ell} \rfloor + 1}{\underset{\uparrow}{k}} \in \{1, \dots, m_0\} : \underset{k\text{th smaller p-value}}{\underset{\uparrow}{p_{(k:\mathcal{H}_0)}}} \leq \frac{\delta k}{m} \Big\}$$

Hence

$$\mathbb{P}(\mathrm{FDP}(\tau_{\hat{\ell}}, P) > \alpha) \leq \mathbb{P}\Big(\exists k \in \{1, \dots, m_0\} : p_{(k:\mathcal{H}_0)} \leq \frac{\delta k}{m_0}\Big) \underset{\uparrow}{\leq} \delta$$

from control of BH(δ) procedure under the full null $\square$

Better result in Gaussian setting with known Γ ?

Reasoning like [Romano Shaikh (2006)] we search a bound on $\mathbb{P}(\mathrm{FDP} > \alpha)$

$$\mathbb{P}(\mathrm{FDP}(\tau_{\hat{\ell}}, P) > \alpha) = \mathbb{P}\left(\sum_{j=1}^{m} (1-\theta_j) \mathbb{1}\{p_j(X) \leq \tau_{\hat{\ell}}\} \geq \lfloor \alpha \hat{\ell} \rfloor + 1\right)$$

$$\leq \sum_{\ell=1}^{m} \mathbb{E}\left[Z_{\ell,\ell} \mathbb{1}\{\hat{\ell} = \ell\}\right]$$

where $Z_{\ell,\ell'} = \mathbb{1}\left\{\sum_{j=1}^{m} \mathbb{1}\{p_j(X-\mu) \leq \tau_\ell\} \geq \lfloor \alpha \ell' \rfloor + 1\right\}$ $\nearrow$ in ℓ $\searrow$ in ℓ'

whose distribution is known !

$$\leq \sum_{\ell=1}^{m} \mathbb{E}[Z_{\ell,\ell} \mathbb{1}\{\hat{\ell} \geq \ell\}] - \underbrace{\sum_{\ell=1}^{m-1} \mathbb{E}[Z_{\ell,\ell} \mathbb{1}\{\hat{\ell} \geq \ell+1\}]}_{= \sum_{\ell=\cancel{2}\, 1}^{m} \mathbb{E}[Z_{\ell-1,\ell-1} \mathbb{1}\{\hat{\ell} \geq \ell\}]}$$

$$= \sum_{\ell=1}^{m} \mathbb{E}[\underbrace{(Z_{\ell,\ell} - Z_{\ell-1,\ell-1})}_{\leq (Z_{\ell,\ell} - Z_{\ell-1,\ell-1})_+} \mathbb{1}\{\hat{\ell} \geq \ell\}]$$

$$\leq \sum_{\ell=1}^{m} \mathbb{E}(Z_{\ell,\ell} - Z_{\ell-1,\ell-1})_+ = B((\tau_\ell)_\ell, \alpha, \zeta, \Gamma)$$ computable bound

[Delattre R. (2015)]

Proposition: in the Gaussian setting with known Γ, choose $\tau_\ell = \zeta \frac{\lfloor \alpha \ell \rfloor + 1}{m}$ and the step-up procedure with critical values $x \times \tau_\ell$, $1 \leq \ell \leq m$

where x the largest such that $B(x(\tau_\ell)_\ell, \alpha, \zeta, \Gamma) \leq \zeta$

Then this procedure controls the FDP

It incorporates dependencies, state of art

Open problem: adaptive FDP control with unknown dependence