Quantitative weak propagation of chaos for stable-driven McKean-Vlasov SDEs

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June 15, 2023

Summer school: mean field systems

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Lévy processes

Lévy process

A $\mathbb{R}^d$-valued process $Z = (Z_t)_{t\geq 0}$ is a Lévy process if

- its paths are a.s. **càdlàg starting from** 0
- it has **stationary increments**, i.e. for any $0 \leq s \leq t$, $Z_t - Z_s$ has the same law as Z_{t-s}
- it has **independent increments**, i.e. for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_d$, $(Z_{t_1}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_d} - Z_{t_{d-1}})$ are independent.

$\hookrightarrow$ Lévy processes are random walks in continuous time.

Example: Brownian motion, Poisson process etc.

Lévy measure

A positive measure ν on $\mathbb{R}^d$ is a Lévy measure if

- $\nu(\{0\}) = 0$.
- $\int_{\mathbb{R}^d} \min(1, |z|^2)\, d\nu(z) < +\infty$.

$\rightarrow$ description of the jumps of Z.

Poisson random measure

Poisson random measure

A random measure $\mathcal{N}$ on $\mathbb{R}^+ \times \mathbb{R}^d$ is a Poisson random measure with intensity $dt \otimes \nu$ if

- $\mathcal{N}([0,t] \times A) \sim \text{Poisson}(t\nu(A))$ for all measurable set A and $t \in \mathbb{R}^+$.
- For all $(A_i)_i$ measurable disjoints, $(\mathcal{N}(A_i))_i$ are independent.

If Z is a Lévy process:

$$\mathcal{N}([0,t] \times A) := \text{Card}\left\{s \in [0,t], \Delta Z_s := Z_s - Z_{s-} \in A\right\}.$$

Compensated Poisson random measure

If A is bounded from below: $\widetilde{\mathcal{N}}([0,t] \times A) := \mathcal{N}([0,t] \times A) - t\nu(A)$. $\widetilde{\mathcal{N}}$ is the compensated Poisson random measure $\rightarrow$ **martingale property**.

α-stable process

α-stable process

A Lévy process Z is a α-stable process with $\alpha \in (0,2]$ if for all $c > 0$, $(Z_{ct})_t$ has the same distribution as $(c^{\frac{1}{\alpha}} Z_t)_t$. **(Scaling property)**

- $\alpha = 2$: Brownian motion.
- Scale: Z_t acts with a typical scale of $t^{\frac{1}{\alpha}}$.
- Moments: **When $\boldsymbol{\alpha < 2}$**, $\forall t$, $\mathbb{E}|Z_t|^\beta < +\infty \Leftrightarrow \beta \in [0, \alpha)$.
- Lévy-Itô decomposition: When $\alpha \in (1,2)$, $Z_t = \int_0^t \int_{\mathbb{R}^d} z \widetilde{\mathcal{N}}(ds, dz) \quad \longrightarrow$ **centered martingale.**
- Lévy measure: writes in polar coordinates $\nu(dz) = \lambda(d\theta)\frac{dr}{r^{1+\alpha}}$, where λ is a non-zero finite measure on the sphere $\mathbb{S}^{d-1}$.

Rotationally invariant α-stable process

Rotationally invariant α-stable process

A α-stable process Z is rotationally invariant if $\nu(dz) = C\frac{dz}{|z|^{d+\alpha}}$.

In the sequel, $C = 1$.

Generator: $\Delta^{\frac{\alpha}{2}}$ given by

$$\Delta^{\frac{\alpha}{2}} f(x) := \int_{\mathbb{R}^d} [f(x+z) - f(x) - \nabla f(x) \cdot z] \, d\nu(z), \quad x \in \mathbb{R}^d$$

$\rightarrow$ non-local operator **(fractional Laplacian)**

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$\rightarrow$ non-local operator **(fractional Laplacian)** defined for $f \in \mathcal{C}_b^{1+\gamma}(\mathbb{R}^d)$ with $\gamma \in (\alpha - 1, 1]$ i.e. $f \in \mathcal{C}_b^1(\mathbb{R}^d)$ and ∇f is γ-Hölder since

$$\int_{D_1} |z|^{1+\gamma}\, d\nu(z) < +\infty \Leftrightarrow \gamma \in (\alpha - 1, 1].$$

McKean-Vlasov SDEs and particle systems

Fix $T > 0$.

$$\begin{cases} dX_t = b(t, X_t, \mu_t)\, dt + dZ_t, \quad t \in [0, T], \\ \mu_t := [X_t], \\ X_0 = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d), \end{cases} \qquad \text{(McKean-Vlasov)}$$

where

- Z rotationally invariant α-stable process with $\alpha \in (1, 2)$,
- $[\xi]$ is the distribution of ξ,
- $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ (Hölder-type assumptions on b).

Mean-field interacting particle system:

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, \overline{\mu}_t^N)\, dt + dZ_t^i, \quad t \in [0, T],\ i \in \{1, \dots, N\}, \\ \overline{\mu}_t^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^{j,N}}, \quad \textbf{(=empirical measure)} \\ X_0^{i,N} = X_0^i, \end{cases} \qquad \text{(Particle system)}$$

where $(Z^i)_i$ and $(X_0^i)_i$ are i.i.d with the same distributions as Z and ξ.

Goal: quantitative weak propagation of chaos

Goal: Prove that $\overline{\mu}_t^N \underset{N\to\infty}{\to} \mu_t$ in distribution **with a rate** using test functions.

Quantitative weak propagation of chaos

Find explicit rates of convergence for $\mathbb{E}|\phi(\overline{\mu}_t^N) - \phi(\mu_t)|$ and $|\mathbb{E}\phi(\overline{\mu}_t^N) - \mathbb{E}\phi(\mu_t)|$, for functions $\phi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ belonging to a sufficiently large class of functions.

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Strategy: use the tools developed with **Mean-Field Games**.

$\hookrightarrow$ Strategy described in the book of Carmona and Delarue inspired by *[Mischler et.al., 2015] and [Cardaliaguet et.al., 2019]*.

$\hookrightarrow$ When Z is a Brownian motion, done by Chaudru de Raynal and Frikha under similar assumptions.

$\hookrightarrow$ Notion of **semigroup** crucial.

Semigroup, generator and PDE: linear case

Standard linear stable-driven SDE:

$$\begin{cases} dX_t^{s,x} = b(t, X_t^{s,x})\, dt + dZ_t, & t \in [s, T], \\ X_s^{s,x} = x, \end{cases}$$

- Semigroup: For $f : \mathbb{R}^d \to \mathbb{R}$, $T_{s,t}f(x) := \mathbb{E}(f(X_t^{s,x})) \to T_{s,t} = T_{s,\tau} \circ T_{\tau,t}$. **(Markov property)**
- Generator: $L_s f(t,x) := b(s,x) \cdot \nabla f(t,x) + \Delta^{\frac{\alpha}{2}} f(t,\cdot)(x)$. **(Itô's formula)**
- Backward Kolmogorov PDE: If b regular enough, $(s,x) \in [0,t] \times \mathbb{R}^d \mapsto T_{s,t}f(x)$ is the unique solution to the parabolic problem

$$\begin{cases} \partial_s u(s,x) + L_s u(s,\cdot)(x) = 0, & \forall (s,x) \in [0,t) \times \mathbb{R}^d, \\ u(t,\cdot) = f. \end{cases} \tag{1}$$

- Transition density: $p(s,t,x,\cdot)$ is the fundamental solution to (1) and $T_{s,t}f(x) = \int_{\mathbb{R}^d} f(y)p(s,t,x,y)\, dy \to$ **Smoothing properties for** (1).

Semigroup, generator and PDE: non-linear case

What happens for (McKean-Vlasov)**?**

McKean-Vlasov SDE:

$$\begin{cases} dX_t^{s,\xi} = b(t, X_t^{s,\xi}, [X_t^{s,\xi}])\, dt + dZ_t, \quad t \in [s, T], \\ X_s^{s,\xi} = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d), \end{cases} \tag{McKean-Vlasov}$$

- **If** (McKean-Vlasov) **well-posed**, $[X_t^{s,\xi}] =: [X_t^{s,\mu}]$ depends on ξ only through its distribution μ.
- Semigroup: For $\phi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, $\mathcal{T}_{s,t}\phi(\mu) := \phi([X_t^{s,\mu}]) \to \mathcal{T}_{s,t} = \mathcal{T}_{s,\tau} \circ \mathcal{T}_{\tau,t}$.

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- Generator: $\mathscr{L}_s$ should be a PDE operator acting on functions defined on the space of measures $\to$ **differentiation of a map w.r.t. the measure + Itô's formula along a flow of measures.**
- Backward Kolmogorov PDE and transition density: **Smoothing properties ?**

Here $\mathcal{P}(\mathbb{R}^d)$ is **infinite-dimensional** while the noise Z is finite-dimensional.
$\hookrightarrow$ **weaker smoothing properties**.

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Differential calculus: linear derivative

$\mathcal{P}_\beta(\mathbb{R}^d)$ = space of probability measure having a finite β-moment $\to$ Wasserstein metric.

Linear derivative

A function $u : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ has a linear derivative if there exists $\frac{\delta}{\delta m} u \in \mathcal{C}^0(\mathcal{P}_\beta(\mathbb{R}^d) \times \mathbb{R}^d; \mathbb{R})$ s.t.

- For all compact $\mathcal{K} \subset \mathcal{P}_\beta(\mathbb{R}^d)$, there exists $C_\mathcal{K} > 0$ s.t.

$$\forall \mu \in \mathcal{K}, \forall v \in \mathbb{R}^d, \left| \frac{\delta}{\delta m} u(\mu)(v) \right| \leq C_\mathcal{K}(1 + |v|^\beta).$$

- For all $\mu, \nu \in \mathcal{P}_\beta(\mathbb{R}^d)$

$$\lim_{h \to 0} \frac{u(\mu + h(\nu - \mu)) - u(\mu)}{h} = \int_{\mathbb{R}^d} \frac{\delta}{\delta m} u(\mu)(v) \, d(\mu - \nu)(v) = \left\langle \frac{\delta}{\delta m} u(\mu), \mu - \nu \right\rangle.$$

$\to$ **Gateaux derivative** in the space of signed measures.

Properties and examples

Example (Linear functions): $u(\mu) = \int_{\mathbb{R}^d} \phi \, d\mu$ with ϕ continuous and $|\phi(x)| \leq C(1 + |x|^\beta)$ then

$$u(\mu) - u(\nu) = \int_0^1 \int_{\mathbb{R}^d} \phi(v) \, d(\mu - \nu)(v) \, d\lambda \Rightarrow \frac{\delta}{\delta m} u(\mu)(v) = \phi(v).$$

Itô's formula for (McKean-Vlasov)

Theorem (Itô's formula along a flow of probability measures)

For the solution to (McKean-Vlasov), *i.e.* $X_t = X_0 + \int_0^t b(s, X_s, \mu_s)\, ds + Z_t$ *(with b bounded), we have*

$$u(\mu_t) - u(\mu_0) = \int_0^t \mathscr{L}_s u(\mu_s)\, ds,$$

where the ***generator*** $\mathscr{L}_s$ *is given by*

$$\mathscr{L}_s u(\mu) := \int_{\mathbb{R}^d} \partial_v \frac{\delta}{\delta m} u(\mu)(v) \cdot b(s, v, \mu)\, d\mu(v)$$
$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} u(\mu)(v+z) - \frac{\delta}{\delta m} u(\mu)(v) - z \cdot \partial_v \frac{\delta}{\delta m} u(\mu)(v) \right] \frac{dz}{|z|^{d+\alpha}}\, d\mu(v).$$

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- Denoting by L_s^μ the generator of the corresponding linear SDE where μ is fixed
$$\mathscr{L}_s u(\mu) = \int_{\mathbb{R}^d} L_s^\mu \frac{\delta}{\delta m} u(\mu)(v)\, d\mu(v).$$
- **Crucial assumption:** $\partial_v \frac{\delta}{\delta m} u(\mu)(\cdot)$ γ-**Hölder** unif. in μ with $\gamma > \alpha - 1$.

Assumptions and example

$$\begin{cases} dX_s^{t,\xi} = b(s, X_s^{t,\xi}, [X_s^{t,\xi}])\,ds + dZ_s, & s \in [t, T], \\ X_t^{t,\xi} = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d), \end{cases} \qquad \text{(McKean-Vlasov)}$$

Assumptions:

- b jointly continuous and globally **bounded** on $[0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$.
- For all t, μ, $b(t, \cdot, \mu)$ is η-**Hölder** on $\mathbb{R}^d$, for some $\eta \in (0, 1]$, unif. in t, μ.
- For all t, x, the map $b(t, x, \cdot)$ has a linear derivative s.t. $\frac{\delta}{\delta m} b(t, x, \mu)(\cdot)$ is η-**Hölder** on $\mathbb{R}^d$ unif. in t, x, μ and $\frac{\delta}{\delta m} b$ is **bounded**.
- For any (t, x, v), the map $\frac{\delta}{\delta m} b(t, x, \cdot)(v)$ has a linear derivative s.t. $\frac{\delta^2}{\delta m^2} b(t, x, \mu)(v, \cdot)$ is η-**Hölder** continuous unif. in t, x, μ, v and $\frac{\delta^2}{\delta m^2} b$ is **bounded**.

Example: $b(t, x, \mu) = \int_{\mathbb{R}^d} \tilde{b}(t, x, y_1, \ldots, y_l)\, d\mu(y_1) \ldots d\mu(y_l)$, with $\tilde{b}$ η-**Hölder** w.r.t. to $(x, y_1, \ldots, y_l)$ unif. in t and **bounded**.

Theorem

The non-linear martingale problem related to (McKean-Vlasov) *is well-posed, thus there exists a unique weak-solution* $(X_s^{t,\mu})_{s\in[t,T]}$.

Object of interest: semigroup acting on the following space

Hölder space

Fix $\beta \in (1,\alpha)$. For $\delta \in [0,1]$, $\mathcal{C}^{(1,\delta)}(\mathcal{P}_\beta(\mathbb{R}^d))$ is the set of functions $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ having a linear derivative s.t.

$$\left| \frac{\delta}{\delta m}\phi(\mu)(v_1) - \frac{\delta}{\delta m}\phi(\mu)(v_2) \right| \leq C|v_1 - v_2|^\delta.$$

For $\phi \in \mathcal{C}^{(1,\delta)}(\mathcal{P}_\beta(\mathbb{R}^d))$, we define

$$U : (t,\mu) \in [0,T] \times \mathcal{P}_\beta(\mathbb{R}^d) \mapsto \phi([X_T^{t,\mu}]).$$

Solution to the backward Kolmogorov PDE with terminal condition

Theorem

The function U belongs to $\mathcal{C}^0([0,T] \times \mathcal{P}_\beta(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T) \times \mathcal{P}_\beta(\mathbb{R}^d))$ *and satisfies:*

$$\left|\partial_v \frac{\delta}{\delta m} U(t,\mu)(v)\right| \leq C(T-t)^{\frac{\delta-1}{\alpha}},$$

and for $\gamma \in (0,1] \cap (0,(2\alpha-2) \wedge (\eta+\alpha-1))$

$$\left|\partial_v \frac{\delta}{\delta m} U(t,\mu)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_2)\right| \leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} |v_1 - v_2|^\gamma.$$

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Moreover, U is the (unique) classical solution to

$$\begin{cases} \partial_t U(t,\mu) + \mathscr{L}_t U(t,\cdot)(\mu) = 0, \quad \forall (t,\mu)\in[0,T)\times\mathcal{P}_\beta(\mathbb{R}^d), \\ U(T,\mu) = \phi(\mu), \quad \forall \mu\in\mathcal{P}_\beta(\mathbb{R}^d), \end{cases}$$

where

$$\mathscr{L}_s h(\mu) := \int_{\mathbb{R}^d} b(s,v,\mu)\cdot\partial_v \frac{\delta}{\delta m} h(\mu)(v)\, d\mu(v) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\frac{\delta}{\delta m}h(\mu)(v+z) - \frac{\delta}{\delta m}h(\mu)(v) - z\cdot\partial_v\frac{\delta}{\delta m}h(\mu)(v)\right]\frac{dz}{|z|^{d+\alpha}}\, d\mu(v).$$

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Notations

Recall that $(\mu_t)_{t\in[0,T]}$ = flow of marginal distributions of (McKean-Vlasov).

Vectorial form for the particle system: We set for $t \in [0, T]$ and $\boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$

$$\boldsymbol{X}_t^N = \begin{pmatrix} X_t^{1,N} \\ \vdots \\ X_t^{N,N} \end{pmatrix}, \quad \boldsymbol{Z}_t^N = \begin{pmatrix} Z_t^1 \\ \vdots \\ Z_t^N \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}^N(t, \boldsymbol{x}) := \begin{pmatrix} b(t, x_1, \overline{\mu}_{\boldsymbol{x}}^N) \\ \vdots \\ b(t, x_N, \overline{\mu}_{\boldsymbol{x}}^N) \end{pmatrix} \in (\mathbb{R}^d)^N$$

$$\begin{cases} d\boldsymbol{X}_t^N = \boldsymbol{b}^N(t, \boldsymbol{X}_t^N)\, dt + d\boldsymbol{Z}_t^N, \quad t \in [0, T], \\ \boldsymbol{X}_0^N = \begin{pmatrix} X_0^1 \\ \vdots \\ X_0^N \end{pmatrix}. \end{cases}$$

The noise $\boldsymbol{Z}^N$ is a cylindrical α-stable process with

- $\boldsymbol{\mathcal{N}}^N$ = Poisson random measure,
- $\boldsymbol{\nu}^N$ = Lévy measure on $(\mathbb{R}^d)^N$.

Quantitative weak propagation of chaos

Recall that $\beta \in (1, \alpha)$.

Test functions

For $\delta \in (0,1]$ and $L > 0$, $\mathcal{C}_L^{(2,\delta)}(\mathcal{P}_\beta(\mathbb{R}^d))$ is the set of functions $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ admitting two linear derivatives $\frac{\delta}{\delta m}\phi$ and $\frac{\delta^2}{\delta m^2}\phi$ s.t.

$$\left|\frac{\delta}{\delta m}\phi(\mu)(v_1) - \frac{\delta}{\delta m}\phi(\mu)(v_2)\right| + \left|\frac{\delta^2}{\delta m^2}\phi(\mu)(v_1, v_1') - \frac{\delta^2}{\delta m^2}\phi(\mu)(v_2, v_2')\right| \leq L(|v_1 - v_2|^\delta + |v_1' - v_2'|^\delta).$$

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$$\left| \frac{\delta}{\delta m}\phi(\mu)(v_1) - \frac{\delta}{\delta m}\phi(\mu)(v_2) \right| + \left| \frac{\delta^2}{\delta m^2}\phi(\mu)(v_1, v_1') - \frac{\delta^2}{\delta m^2}\phi(\mu)(v_2, v_2') \right| \leq L(|v_1 - v_2|^\delta + |v_1' - v_2'|^\delta).$$

Theorem

Let us fix $\gamma \in (0,1] \cap (0, (\delta + \alpha - 1) \wedge (2\alpha - 2) \wedge (\eta + \alpha - 1))$. There exists a positive constant $C_T = C(d, T, \alpha, \beta, \eta, \gamma, \delta, L)$ non-decreasing in T such that for all $\phi \in \mathcal{C}_L^{(2,\delta)}(\mathcal{P}_\beta(\mathbb{R}^d))$, it holds

$$\mathbb{E}|\phi(\overline{\mu}_T^N) - \phi(\mu_T)| \leq C_T \mathbb{E} W_1(\overline{\mu}_0^N, \mu_0) + \frac{C_T}{N^{1-\frac{1}{\beta}}},$$

and

$$|\mathbb{E}(\phi(\overline{\mu}_T^N) - \phi(\mu_T))| \leq \frac{C_T}{N^\gamma}.$$

Remark: For all $\varphi : \mathbb{R}^d \to \mathbb{R}$ s.t. $\|\varphi\|_{\text{Lip}} \leq 1$ i.e. $|\varphi(x) - \varphi(y)| \leq |x - y|$, define $\phi(\mu) := \int_{\mathbb{R}^d} \varphi \, d\mu$. Then $\phi \in \mathcal{C}_1^{(2,1)}(\mathcal{P}_\beta(\mathbb{R}^d))$.

$\hookrightarrow$ **Rate of convergence w.r.t. W_1.** We have by the Kantorovich-Rubinstein theorem

$$\begin{aligned} W_1([X_T^{1,N}], \mu_T) &= \sup_{\varphi,\, \|\varphi\|_{\text{Lip}} \leq 1} \left| \mathbb{E}\varphi(X_T^{1,N}) - \int_{\mathbb{R}^d} \varphi \, d\mu_T \right| \\ &= \sup_{\varphi,\, \|\varphi\|_{\text{Lip}} \leq 1} \left| \mathbb{E} \int_{\mathbb{R}^d} \varphi \, d\overline{\mu}_T^N - \mathbb{E} \int_{\mathbb{R}^d} \varphi \, d\mu_T \right| \\ &= \sup_{\phi \in \mathcal{C}_1^{(2,1)}(\mathcal{P}_\beta(\mathbb{R}^d))} \left| \mathbb{E}\phi(\overline{\mu}_T^N) - \mathbb{E}\phi(\mu_T) \right| \\ &\leq \frac{C}{N^\gamma}. \end{aligned}$$

$\rightarrow$ Allows also **non-linear functions** of the empirical measure.

Idea of the proof

- Consider $U(t,\mu) = \phi([X_T^{t,\mu}])$ the solution to the Kolmogorov PDE.
- Compare $\phi(\overline{\mu}_T^N) - \phi(\mu_T) = U(T,\overline{\mu}_T^N) - U(T,\mu_T)$ through dynamics of $t \in [0,T) \mapsto U(t,\overline{\mu}_t^N) - U(t,\mu_t)$.
- The map $t \in [0,T] \mapsto U(t,\mu_t) = \phi([X_T^{t,\mu_t}])$ is constant by the **flow property**.

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- The map $t \in [0,T] \mapsto U(t,\mu_t) = \phi([X_T^{t,\mu_t}])$ is constant by the **flow property**.

$\to$ Additional regularity on U: for $\gamma \in (0,1] \cap (0, (2\alpha-2) \wedge (\eta+\alpha-1))$

$$\left|\partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu_2)(v_2)\right| \leq C \underbrace{(T-t)^{\frac{\delta-1-\gamma}{\alpha}}}_{\textbf{integrable for } \gamma < \delta+\alpha-1} \left(|v_1-v_2|^\gamma + W_1^\gamma(\mu_1,\mu_2)\right).$$

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- Consider $U(t,\mu)=\phi([X_T^{t,\mu}])$ the solution to the Kolmogorov PDE.
- Compare $\phi(\overline{\mu}_T^N)-\phi(\mu_T)=U(T,\overline{\mu}_T^N)-U(T,\mu_T)$ through dynamics of $t\in[0,T)\mapsto U(t,\overline{\mu}_t^N)-U(t,\mu_t)$.
- The map $t\in[0,T]\mapsto U(t,\mu_t)=\phi([X_T^{t,\mu_t}])$ is constant by the **flow property**.

$\rightarrow$ Additional regularity on U: for $\gamma\in(0,1]\cap(0,(2\alpha-2)\wedge(\eta+\alpha-1))$

$$\left|\partial_v\frac{\delta}{\delta m}U(t,\mu_1)(v_1)-\partial_v\frac{\delta}{\delta m}U(t,\mu_2)(v_2)\right|\leq C\underbrace{(T-t)^{\frac{\delta-1-\gamma}{\alpha}}}_{\textbf{integrable for }\boldsymbol{\gamma<\delta+\alpha-1}}\left(|v_1-v_2|^\gamma+W_1^\gamma(\mu_1,\mu_2)\right).$$

$\rightarrow$ Apply Itô's formula for the **empirical projection** $(t,\boldsymbol{x})\in[0,T)\times(\mathbb{R}^d)^N\mapsto U(t,\overline{\mu}_{\boldsymbol{x}}^N)$, where $\overline{\mu}_{\boldsymbol{x}}^N=\frac{1}{N}\sum_{k=1}^N\delta_{x_k}$ and for the particle system using that

$$\partial_{x_i}U(t,\overline{\mu}_{\boldsymbol{x}}^N)=\frac{1}{N}\partial_v\frac{\delta}{\delta m}U(t,\overline{\mu}_{\boldsymbol{x}}^N)(x_i)\quad\rightarrow\textbf{choose }\ \boldsymbol{\gamma>\alpha-1}.$$

Idea of the proof

$$
\begin{aligned}
&U(t,\overline{\mu}_t^N) - U(t,\mu_t) - \left(U(0,\overline{\mu}_0^N) - U(0,\mu_0)\right) \\
&= \int_0^t \partial_t U(s,\overline{\mu}_s^N)\,ds \\
&\quad + \frac{1}{N}\sum_{i=1}^N \int_0^t \partial_v \frac{\delta}{\delta m} U(s,\overline{\mu}_s^N)(X_s^{i,N}) \cdot b(s,X_s^{i,N},\overline{\mu}_s^N)\,ds \\
&\quad + \int_0^t \int_{(\mathbb{R}^d)^N} \left[U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}+\mathbf{z}}) - U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}}) - \partial_{\mathbf{x}} U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}}) \cdot \mathbf{z} \right] d\boldsymbol{\nu}^N(\mathbf{z})\,ds \\
&\quad + \int_0^t \int_{\{|\mathbf{z}|\geq 1\}} \left[U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}+\mathbf{z}}) - U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds,d\mathbf{z}) \\
&\quad + \int_0^t \int_{|\mathbf{z}|<1\}} \left[U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}+\mathbf{z}}) - U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds,d\mathbf{z}) \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

→ **Goal:** make appear the Kolmogorov PDE.

Idea of the proof

→ Note that I_2 can be rewritten as

$$I_2 = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \partial_v \frac{\delta}{\delta m} U(s,\overline{\mu}_s^N)(X_s^{i,N}) \cdot b(s,v,\overline{\mu}_s^N)\, ds$$
$$= \int_0^t \int_{\mathbb{R}^d} \partial_v \frac{\delta}{\delta m} U(s,\overline{\mu}_s^N)(v) \cdot b(s,v,\overline{\mu}_s^N)\, d\overline{\mu}_s^N(v)\, ds.$$

This is one term of the PDE satisfied by U.

→ Concerning I_3:

$$I_3 = \int_0^t \int_{(\mathbb{R}^d)^N} \left[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}}) - \partial_{\boldsymbol{x}} U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}}) \cdot \boldsymbol{z} \right] d\boldsymbol{\nu}^N(\boldsymbol{z})\, ds$$

does not appear in the PDE satisfied by U (only terms involving the linear derivative of U).

↪ **Linearisation** with the linear derivative.

Idea of the proof

Setting $m^i_{s,z,w} := w\overline{\mu}^N_{\boldsymbol{X}^N_{s-}+\tilde{\boldsymbol{z}}_i} + (1-w)\overline{\mu}^N_{\boldsymbol{X}^N_{s-}}$

$$
\begin{aligned}
I_3 = \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d}\int_0^1 \Big[& \frac{\delta}{\delta m}U(s, m^i_{s,z,w})(X^{i,N}_{s-}+z) \\
& - \frac{\delta}{\delta m}U(s, m^i_{s,z,w})(X^{i,N}_{s-}) - \partial_v \frac{\delta}{\delta m}U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s-}})(X^{i,N}_{s-})\cdot z\Big]\, dw\, d\nu(z)\, ds, \\
= \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big[& \frac{\delta}{\delta m}U(s, \overline{\mu}^N_{s-})(x+z) - \frac{\delta}{\delta m}U(s, \overline{\mu}^N_{s-})(x) \\
& - \partial_v \frac{\delta}{\delta m}U(s, \overline{\mu}^N_{s-})(x)\cdot z\Big]\, d\nu(z)\, d\overline{\mu}^N_{s-}(x)\, ds \\
& + \text{Error term.}
\end{aligned}
$$

$\hookrightarrow$ Allows to use the PDE satisfied by U.

Idea of the proof

Using the PDE, it remains

$$
\begin{aligned}
&U(t,\overline{\mu}_t^N) - U(t,\mu_t) - \left(U(0,\overline{\mu}_0^N) - U(0,\mu_0)\right) \\
&= \text{Error term} \\
&\quad + \int_0^t \int_{\{|\mathbf{z}|\geq 1\}} \left[U(s,\overline{\mu}^N_{\mathbf{X}^N_{s^-}+\mathbf{z}}) - U(s,\overline{\mu}^N_{\mathbf{X}^N_{s^-}}) \right] \widetilde{\mathcal{N}}^N(ds,d\mathbf{z}) \\
&\quad + \int_0^t \int_{\{|\mathbf{z}|<1\}} \left[U(s,\overline{\mu}^N_{\mathbf{X}^N_{s^-}+\mathbf{z}}) - U(s,\overline{\mu}^N_{\mathbf{X}^N_{s^-}}) \right] \widetilde{\mathcal{N}}^N(ds,d\mathbf{z}) \\
&= \text{Error term} + I_4 + I_5.
\end{aligned}
$$

- Use **martingale property/BDG's inequalities** for I_4 and I_5.
- Find upper-bounds for each term using **the bounds/Hölder controls** on the derivatives w.r.t. the measure of U $\rightarrow$ **time-integrability of the singularities.**
- Conclude using the continuity of U that

$$
U(t,\overline{\mu}_t^N) - U(t,\mu_t) \underset{t\to T}{\longrightarrow} U(T,\overline{\mu}_T^N) - U(T,\mu_T) = \phi(\overline{\mu}_T^N) - \phi(\mu_T) \quad a.s.
$$

Thank you for your attention !

References:

- *Itô's formula for the flow of measures of Poisson stochastic integrals and applications*, T.C. 2022.

- *Quantitative weak propagation of chaos for stable-driven McKean-Vlasov SDEs*, T.C. 2022.