The number of critical points of a Gaussian field: finiteness of moments

Louis GASS
joint work with Michele Stecconi

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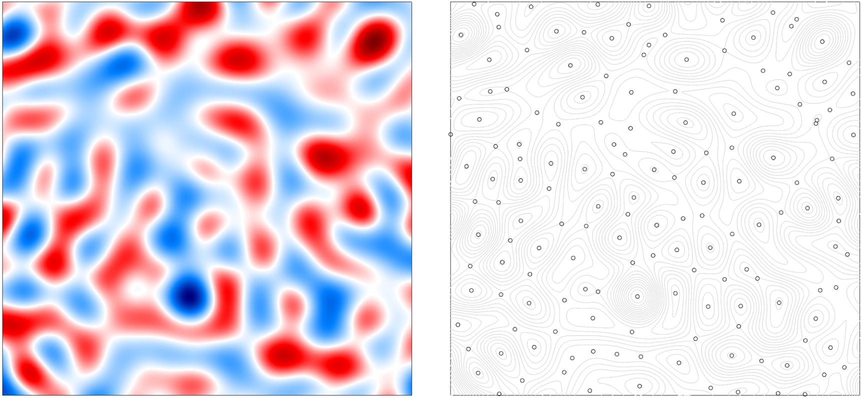
Random fields and critical points



Simulation of a planar random field and its critical points.

Outline

Introduction and motivations

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth random field. Let

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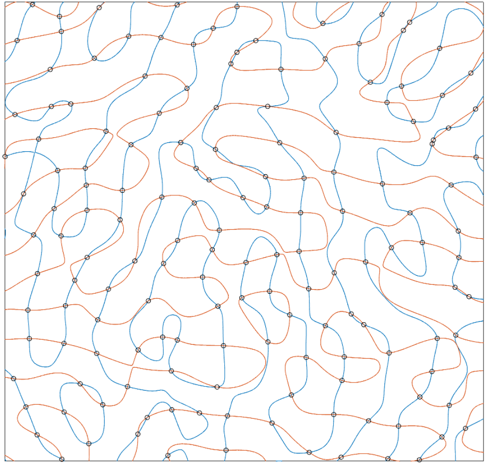

Zero sets of the two partial derivatives of a planar Gaussian process

The moment conjecture

Conjecture

Assume that the covariance function of the Gaussian field f and its derivatives are in $L^2(\mathbb{R}^d)$. Then for every integer $p \geq 1$,

$$\lim_{R\to+\infty} \mathbb{E}\left[\left(\frac{N_R(f) - \mathbb{E}[N_R(f)]}{\sqrt{\mathrm{Var}(N_R(f))}}\right)^p\right] = \mathbb{E}[W^p],$$

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- Finiteness of moments:

$\to$ Cuzick (1975)

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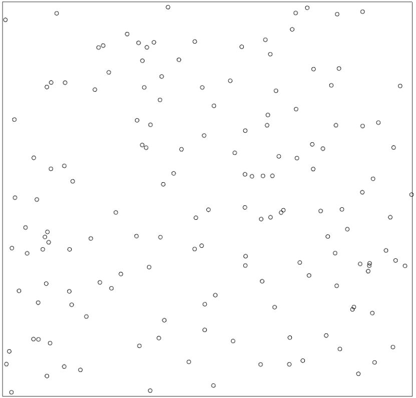
$\rightarrow$ Nazarov–Sodin (2012)

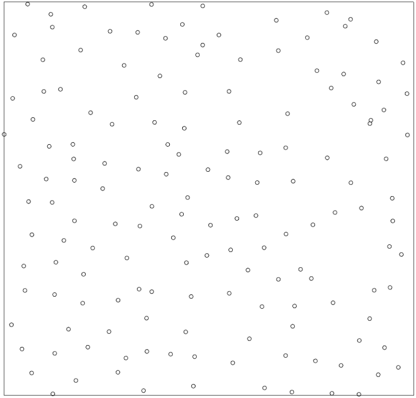
$\rightarrow$ Ancona–Letendre (2020)

$\rightarrow$ G. (2022)

Spatial distribution of critical points



Homogeneous Poisson point process



Critical points of Gaussian process

Spatial distribution of critical points

Study of 2-points intensity function :

Theorem (Azaïs–Delmas (2019))

There is

- *repulsion of critical points when* $d = 1$,
- *neutrality of critical points when* $d = 2$,
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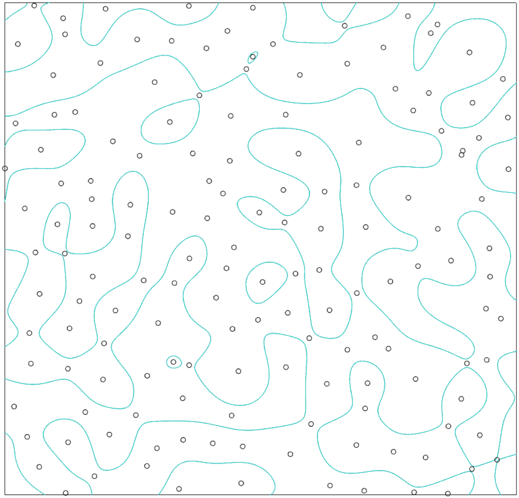
$\rightarrow$ 2-points intensity function does not explain the apparent rigidity.

Critical points and connected components



random nodal set and critical points

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$\to$ No result for moments of order $p \geq 4$ in dimension $d \geq 2$.

Main result

Theorem (G.–Stecconi (2023))

Let $f : \mathbb{R}^d \to \mathbb{R}$ *be a Gaussian process of class* $\mathcal{C}^{p+1}$. *Assume that*

$$\forall x \in B(0,R), \quad \det \mathrm{Cov}\left((\partial^\alpha f(x))_{|\alpha| \le p+1}\right) > 0.$$

Then

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$\rightarrow$ Extend the previous result of Beliaev–Mcauley–Muirhead to any p.

Kac–Rice formula

Let

$$\Delta = \left\{ \underline{x} \in (\mathbb{R}^d)^p \,\middle|\, \exists i \neq j \ \text{ s.t. } \ x_i = x_j \right\}.$$

Theorem (Kac–Rice formula)

Let f be a process of class $\mathcal{C}^2$ such that $(\nabla f(x_i))_{1 \leq i \leq p}$ has a density $\psi_{\underline{x}}^f$ for all $\underline{x} \in B(0,R)^p \setminus \Delta$. Then

$$\mathbb{E}[N_R(f)^{[p]}] = \int_{B(0,R)^p \setminus \Delta} \rho_f(\underline{x}) \mathrm{d}\underline{x},$$

where

$$\rho_f(\underline{x}) = \mathbb{E}\left[\prod_{k=1}^{p} |\det \mathrm{Hess} f(x_k)| \,\middle|\, \nabla f(x_1) = \ldots = \nabla f(x_p) = 0 \right] \psi_{\underline{x}}^f(0).$$

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$\to$ Difficult to understand the behavior of ρ_f near the diagonal Δ.

Key observation

In the following f is a C^{p+1} Gaussian field such that

$$\forall x \in B(0,R), \quad \det \operatorname{Cov}\left((\partial^\alpha f(x))_{|\alpha|\leq p+1}\right) > 0.$$

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For R small enough and all $\underline{x} \in B(0,R)^p \setminus \Delta$,

$$\rho_f(\underline{x}) = Q(\underline{x})\sigma_f(\underline{x}),$$

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- *Q is universal (does not depend on f)*
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$$\rho_f \leq \frac{\sup \sigma_f}{\inf \sigma_g}\, \rho_g \in L^1(B(0,R)).$$

Extracting the singularity

$$\rho_f(\underline{x}) = \frac{\mathbb{E}\left[\prod_{k=1}^{p} |\det \mathrm{Hess} f(x_k)| \,\middle|\, \nabla f(x_1) = \ldots = \nabla f(x_p) = 0\right]}{\sqrt{\det 2\pi \mathrm{Cov}\left(\nabla f(x_1), \ldots, \nabla f(x_p)\right)}}.$$

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$\rightarrow$ No *well-poised* interpolation in higher dimensions (Mairhuber–Curtis)

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$$\delta_{\underline{x}} = \begin{pmatrix} \delta_{x_1} \\ \delta_{x_2} \\ \vdots \\ \delta_{x_p} \end{pmatrix} = A(\underline{x}) \begin{pmatrix} \frac{\delta_{x_1}}{\|\delta_{x_1}\|} \\ \frac{\delta_{x_2} - \mathrm{Proj}_{\delta_{x_1}}(\delta_{x_2})}{\|\delta_{x_2} - \mathrm{Proj}_{\delta_{x_1}}(\delta_{x_2})\|} \\ \vdots \\ \frac{\delta_{x_p} - \mathrm{Proj}_{\mathrm{Span}(\delta_{x_1}, \ldots, \delta_{x_{p-1}})}(\delta_{x_p})}{\|\delta_{x_p} - \mathrm{Proj}_{\mathrm{Span}(\delta_{x_1}, \ldots, \delta_{x_{p-1}})}(\delta_{x_p})\|} \end{pmatrix}$$

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Evaluating at a function f:

$$\delta_{\underline{x}} f = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_p) \end{pmatrix} = A(\underline{x}) \begin{pmatrix} f(x) \\ f[x, y] \\ \vdots \\ f[x_1, \ldots, x_p] \end{pmatrix}$$

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- $Q_0(\underline{x})$ is a universal square matrix of size dp,
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$$\sqrt{\det \operatorname{Cov}\left(\nabla f(x_1), \dots, \nabla f(x_p)\right)} = |\det Q_0(\underline{x})| \sqrt{\det \operatorname{Cov}(N_f(\underline{x}))}.$$

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Similarly,

$$\mathbb{E}\left[\prod_{k=1}^{p} |\det \mathrm{Hess} f(x_k)| \middle| \nabla_{\underline{x}} f = 0\right] = \left(\prod_{k=1}^{p} Q_k(\underline{x})\right) \mathbb{E}\left[\prod_{k=1}^{p} |H_k(\underline{x})| \middle| N_f(\underline{x}) = 0\right],$$

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$\to$ The Kac–Rice decomposition is achieved

It remains to show that:

- there is an adequate scalar product for evaluation maps
- the function σ_f is bounded above and below by positive constants.

Kergin interpolation

Theorem (Kergin (1980))

For $\underline{x} = (x_0, x_1, \ldots, x_p) \in (\mathbb{R}^d)^{p+1}$ *there is a projector*

$$\Pi_{\underline{x}} : \mathcal{C}^p(\mathbb{R}^d) \to \mathbb{R}_p[X_1, \ldots, X_d]$$

such that if the multiplicity of x_k *in* $\underline{x}$ *is* n *then*

$$\forall |\alpha| < n, \quad \partial^\alpha \left(\Pi_{\underline{x}} f\right)(x_k) = \partial^\alpha f(x_k).$$

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$\to$ We see $\nabla_{\underline{x}}$ as a family of linear forms on a finite dimensional space

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Theorem (Kergin (1980))

For $\underline{x} = (x_0, x_1, \ldots, x_p) \in (\mathbb{R}^d)^{p+1}$ *there is a projector*

$$\Pi_{\underline{x}} : \mathcal{C}^p(\mathbb{R}^d) \to \mathbb{R}_p[X_1, \ldots, X_d]$$

such that if the multiplicity of x_k *in* $\underline{x}$ *is* n *then*

$$\forall |\alpha| < n, \quad \partial^\alpha \left(\Pi_{\underline{x}} f\right)(x_k) = \partial^\alpha f(x_k).$$

The polynomial $\Pi_{\underline{x}} f$ **does not depend only** on $(f(x_1), \ldots, f(x_p))$.

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Boundedness of σ_f follows from:

- **Compactness** properties of $\mathbb{R}_p[X_1, \ldots, X_d]$
- **Non-degeneracy** of the $p+1$ jets of f

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- Exponential moment for analytic fields
  $\to$ Exponential concentration of nodal volume

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