GRAPH COMPLEXES

GEOFFREY POWELL

ABSTRACT. Graph complexes are fascinating mathematical objects; they are complexes (in the sense of homological algebra) and the associated homology is called graph homology.

In their simplest guise, they have a straightforward combinatorial definition: terms are linear combinations of graphs and the differential is induced by edge contraction (or, dually, by edge expansion). However, whilst simple to define, the calculation of graph homology is very difficult in general.

There are variants; for instance, vertexes can be decorated, edges can be oriented, and one can introduce degrees on edges and/ or vertexes. One can also add 'hairs', by marking vertexes, leading to hairy graph complexes and their homology.

Graph homology is related to deep mathematical structures, as was shown by Kontsevich in the early 1990s. He considered three flavours of graph complexes, related to the commutative, the Lie, and the associative worlds (referring to types of algebras implicitly used in their construction). On the one hand, Kontsevich showed that the associated graph homology is isomorphic to the homology of certain infinite-dimensional Lie algebras; on the other, he showed that the homology calculates interesting invariants of fundamental geometric structures. For instance, in the Lie world, the homology is related to that of outer automorphism groups of free groups; in the associative world, the homology is related to that of the moduli space of curves.

Hairy graph homology is also known to have deep geometrical significance; for instance, in the commutative world, it calculates the rational homotopy groups of certain embedding spaces.

Their are more sophisticated viewpoints. For instance, Getzler and Kapranov placed the construction of graph complexes in the wider context of their Feynman transform applied to *modular operads*. Moreover, graph complexes can be interpreted as *deformation complexes* in *operad theory*.

The aim is to introduce graph complexes, their homology, and some properties, also outlining some applications. It is hoped that this will serve as an entry point to more sophisticated aspects of the theory and recent research.