Replication of arithmetic random waves

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Helmholtz equation

Eigenmodes: Solutions F_k of

$$\Delta F + k^2 F = 0$$

- Δ: Laplacian operator on a manifold (here $\mathbb{R}^2$ or $\mathbb{T}^2$)
- k: wavenumber
- Spatial component of solutions of d'Alembert wave propagation equation
- On $\mathbb{R} : F_k(x) = a\cos(kx) + b\sin(kx)$
- On $\mathbb{R}^2$: for $u \in \mathbb{R}^2, \|u\| = k$

$$F_u(x) = \cos(\langle u, x\rangle) \text{ or } \sin(\langle u, x\rangle)$$
$$+ \text{ linear combinations}$$

Eigenmodes on $\mathbb{T}^2$

$$F_u(x) = \cos(\langle u, x\rangle) \text{ or } \sin(\langle u, x\rangle)$$

- F_u continuous on $\mathbb{T}^2$ $\Leftrightarrow$ F_u is $(1,1)$-periodic $\Leftrightarrow$ $u \in \mathbb{Z}^2$
- $\Delta F_u(x) = -4\pi^2(u_1^2 + u_2^2)F_u(x) = -4\pi^2\|u\|^2 F_u(x)$
- For $n \in \mathbb{N}$, the nth eigenspace is generated by

$$\mathcal{E}_n = \{F_u : \|u\|^2 = n\} \text{ (Solutions of } \Delta F + 4\pi^2 nF = 0)$$

- In particular, n has to be written as the sum of two squares.

$$\mathscr{S} := \{n : \mathcal{E}_n \neq 0\}$$

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- A prime number p is the sum of two squares if $p = 2$ or $p \equiv 1 \mod 4$, in this case

$$p = a_p^2 + b_p^2 = (a_p + ib_p)(a_p - ib_p)$$

- **General case**: $n \in \mathscr{S}$ if

$$n = p_1^{\alpha_1} \dots p_m^{\alpha_m} q_1^{2\beta_1} \dots . q_l^{2\beta_l}$$

with $p_i = 2$ or $p_i \equiv 1, q_i \equiv 3$. Several solutions z_j

$$n = z_j \bar{z}_j$$

$$z_j = \prod_i (a_{p_i} \pm ib_{p_i})^{\alpha_i} \times \underbrace{Z_n}_{q_1^{\beta_1} \dots . q_l^{\beta_l}}$$

- **Cardinality**

$$\mathcal{N}_n := \#\mathcal{E}_n = \begin{cases} 0 \text{ if some } q_i \text{ has odd valuation} \\ 4 \prod_{i=1}^m (1 + \alpha_i) \text{ otherwise} \end{cases}$$

Arithmetic Random waves

- For **most** $n \in \mathscr{S}$

$$\mathcal{N}_n = \ln(n)^{\ln(2)/2+o(1)}$$

(i.e. for a density 1 subsequence $\mathscr{S}'$ of integers $n \subset \mathscr{S}$),

- Let $F_n : \sqrt{n}\mathbb{T}^2 \to \mathbb{R}$ the **Planck scale** Arithmetic Random Wave (ARW):

$$F_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{u:\|u\|^2=n} \left[a_u \cos\left(\left\langle x, \frac{u}{\sqrt{n}} \right\rangle\right) + b_u \sin\left(\left\langle x, \frac{u}{\sqrt{n}} \right\rangle\right)\right]$$

- The **covariance function** is for $x, y \in \sqrt{n}\mathbb{T}^2$

$$\begin{aligned} r_n(x-y) &= \mathrm{Cov}(F_n(x), F_n(y)) = \mathbb{E}\left[F_n(x)F_n(y)\right] \\ &= \frac{1}{\mathcal{N}_n} \sum_{u\in\mathbb{Z}^2:\|u\|^2=n} \cos\left(\left\langle x-y, \frac{u}{\sqrt{n}} \right\rangle\right). \end{aligned}$$

Convergence of the covariance function

$$r_n(x) = \frac{1}{\mathcal{N}_n} \sum_{u \in \mathbb{Z}^2 : \|u\|^2 = n} \cos(\langle x, u/\sqrt{n} \rangle) = \int_{\mathbb{S}^1} \cos(\langle x, u \rangle) d\mu_n(u)$$

where $\mu_n := \frac{1}{\mathcal{N}_n} \sum_{u \in \mathbb{Z}^2 : \|u\|^2 = n} \delta_{\frac{u}{\sqrt{n}}} \xrightarrow[n \to \infty]{} \mu_{\mathbb{S}^1}$ Haar measure on $\mathbb{S}^1$

for $n \in \mathscr{S}'' \subset \mathbb{N}$ of density 1. **Pointwise convergence** to the $0-$Bessel function

$$r_n(x) \to J_0(x) = \int \cos(\langle x, u \rangle) d\mu_{\mathbb{S}^1}(u)$$

Remark: J_0 is the covariance function of an isotropic stationary field F_∞ on $\mathbb{R}^2$, the **Random planar wave model**:

$$\mathsf{Cov}(F_\infty(x), F_\infty(y)) = J_0(x - y)$$

Berry's conjecture on nodal lines

Expectation: **Oravecz, Rudnick and Wigman '08**

$$\mathscr{L}_B := \text{length}\{F_n^{-1}(\{0\}) \cap B\}, B \subset \sqrt{n}\mathbb{T}^2$$

$$\mathbb{E}(\mathscr{L}_B) = |B| \frac{1}{2\sqrt{2}}$$

Variance: **Krishnapur, Kurlberg, Wigman 2011** : For $n \in \mathscr{S}'$

$$\text{Var}(\mathscr{L}_{\sqrt{n}\mathbb{T}^2}) \sim \frac{c_n}{512} \frac{n^2}{\mathcal{N}_n^2} \text{ where } c_n \in [1/2, 1] \text{ "oscillates" as } n \to \infty$$

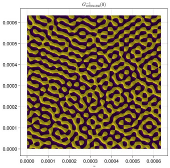

Figure: Nodal lines (L. Thomassey)

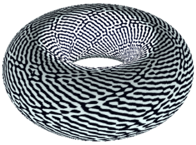

Figure: Excursion (Simon Coste)

Small balls and full correlation

- Generalisation by **Benatar, Marinucci, Wigman 2020** to small balls: For $\alpha > 0, s_n > n^\alpha$,

$$\mathscr{L}_{s_n} := \mathrm{length}\{F_n^{-1}(\{0\}) \cap \mathsf{B}(s_n)\} \ \ \mathsf{Var}(\mathscr{L}_{s_n}) \sim c_n |\mathsf{B}(s_n)|^2 \frac{1}{\mathcal{N}_n^2}$$

- Furthermore, there is **full correlation** between small balls and $\sqrt{n}\mathbb{T}^2$:

$$\sup_{s \geqslant n^\alpha} |\mathsf{Corr}(\mathscr{L}_s, \mathscr{L}_{\sqrt{n}\mathbb{T}^2}) - 1| \to 0.$$

- Based on the **Kac-Rice formula** and computations of the **spectral quasi-correlations**

$$\#\{(u_1, \ldots, u_l) \in (\mathbb{Z}^2)^l : 0 < |u_1 + \cdots + u_l| < \varepsilon, \|u_i\|^2 = n\}$$

- Interpretation in **[Todino 2020]** (no full correlation on $\mathbb{S}^2$)

Phase transition

- There is full correlation at **polynomial scales** [BMW 20'].
  Furthermore

$$\widetilde{\mathscr{L}}_{n^\alpha} := \frac{\mathscr{L}_{n^\alpha} - \mathbb{E}(\mathscr{L}_{n^\alpha})}{\sqrt{\mathrm{Var}}} \to \text{sum of Chi}^2 \text{ variables}$$

- Drastic change of behaviour at **logarithmic scales** [**Dierickx, Nourdin, Peccati and Rossi '19**]

$$\widetilde{\mathscr{L}}_{\ln(n)^A} \to \mathcal{N}(0,1) \text{ for } A \leqslant \frac{1}{18}\ln(\pi/2)$$

- There are conjectures about the phase transition, i.e. the minimal scale $\ln(n)^{A_c}$ where full correlation occurs:
  - [**Sartori '21**] Full correlation for $s_n = \ln(n)^B$ with $B = \frac{29}{6}\ln(2)$
  - Hence $A < A_c < B$

What happens above the phase transition?

- Intuitively, the nodal lines **replicate** almost identically at distance $\ln(n)^A$ $(A > A_c)$.
- Say that τ is an ε-**almost period** of a function $F : \mathbb{R}^d \to \mathbb{R}^k$ if

$$\sup_t \|F(t+\tau) - F(t)\| < \varepsilon$$

- A function $F : \mathbb{R}^d \to \mathbb{R}$ is **almost periodic** if for all $\varepsilon > 0$ there is a **relatively denset set** of ε-periods.
- A sequence of functions $(F_n)_{n\geqslant 1}$ is said to be $(t_n)_{n\geqslant 1}$-**almost periodic** for some τ_n with $1 \leqslant \|\tau_n\| \leqslant t_n$ if

$$\sup_t \|F_n(t+\tau_n) - F_n(t)\| \to 0.$$

- The (Planck scale) ARW are trivially $(\sqrt{n})$-(almost) periodic.

$\to$ **Are the ARW $(\ln(n)^A)_{n\geqslant 1}$-almost periodic?**

Are the ARW $(\ln(n)^A)_{n\geqslant 1}$-almost periodic?

Theorem (Thomassey, L. 23+)

The covariance function is almost periodic at **intermediates scales***: there is an* **almost period** τ_n *such that asymptotically for* $\alpha > 0$

$$\ln(n)^A \ll \|\tau_n\| \ll n^\alpha$$

$$\text{....actually } \|\tau_n\| = O(\underbrace{\exp(\ln(n)^{\ln(2)/2+}}_{\exp(\mathcal{N}_n^{1+})})$$

and the ARW and its derivatives are (τ_n)*-almost periodic : for* β *any multi-index, with high probability*

$$\sup_{t\in\sqrt{n}\mathbb{T}^2} |\partial^\beta F_n(t) - \partial^\beta F_n(t+\tau_n)| = o(\ln(n)^{-\delta}), \delta > 0$$

Remark: Much smaller than the actual exact period $\sqrt{n}$.

Almost periodicity

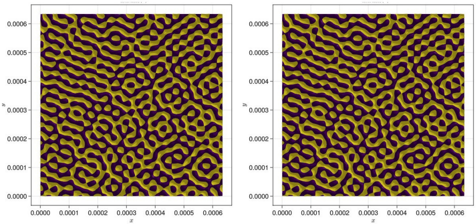

Figure: $n = 10^9$: Game of the 7 differences between F_n and $F_n^{\tau_n} = F_n(\tau_n + \cdot)$

Proof

1. Show that $r(\tau_n) > 1 - \exp(-\ln(n)^{0+})$ (Dirichlet principle)
2. Use concentration results about suprema of random Gaussian fields

$$\sup_{x \in \sqrt{n}\mathbb{T}^2} |F_n - F_n^{\tau_n}|.$$

Consequences for nodal sets

- **Geometric similarity**: for φ continuous with compact support,

$$\int_{F_n^{-1}(\{0\})} \varphi(t)\mathcal{H}^1(dt) - \int_{F_n^{-1}(\{0\})} \varphi(t+\tau_n)\mathcal{H}^1(dt) \xrightarrow[n\to\infty]{\mathscr{L}} 0$$

Proof.

First prove the convergence in law in $\mathcal{C}^2(\overline{Supp(\varphi)} \times \overline{Supp(\varphi)})$

$$(F_n, F_n^{\tau_n}) \to (F_\infty, F_\infty)$$

where F is the planar RPW model on $\mathbb{R}^2$, for the topology of $\mathcal{C}^2$ uniform convergence on each compact, and then prove the continuity of the mapping

$$F \to \int_{F^{-1}(0)} \varphi(t)\mathcal{H}^1(dt)$$

To do list:

- Do we have with high probability

$$\text{Topology}(F_n^{-1}(\{0\}) \cap B) \sim \text{Topology}(F_n^{-1}(\{0\}) \cap (B + \tau_n))?$$

- Replication of *phase singularities*, i.e. (isolated) complex zeros of

$$F_n + iF_n'$$

where F_n' is an independent copy of F_n?

Almost periods of trigonometric polynomials

Lemma

Let $N > 1$ *and*

$$r(x) = \frac{1}{N}\sum_{k=1}^{N} \gamma_k(2\pi\langle u_k, x\rangle), x \in \mathbb{R}^d$$

where the γ_k *are* 1*-Lipschitz and* 2π*-periodic, and* $u_k \in \mathbb{R}^d$*. Then for* $\varepsilon > 0,$ *for some* $1 \leqslant \|\tau\| \leqslant \varepsilon^{-N/d}$*,*

$$|r(t+\tau) - r(t)| \leqslant c\varepsilon \quad (c\varepsilon\text{-almost periodic at scale } \tau)$$

Application to ARW

- $d = 2, N = \mathcal{N}_n = \ln(n)^{\ln(2)/2+o(1)}$
- $u_k \in \mathbb{Z}^2$ such that $\|u_k\|^2 = n$,
- $\tau_n^{\max}$ cannot be logarithmic if $\varepsilon_n \to 0$

$$\varepsilon_n{}^{-\mathcal{N}_n/d} = \tau_n^{\max} \Leftrightarrow \ln \varepsilon_n = \frac{-d\ln(\tau_n^{\max})}{\ln(n)^{\frac{\ln(2)}{2}+o(1)}} \xrightarrow[n\to\infty]{} -\infty?$$

- $\varepsilon_n = \exp(-\ln(n)^{0+}) = \exp(-\mathcal{N}_n^{1+}/\mathcal{N}_n) \Rightarrow \tau \leqslant \exp(\mathcal{N}_n^{1+}/d)$

Lower bound

- [**Dierickx, Nourdin, Peccati and Rossi '19**]: $r_n \to J_0$ uniformly on $\mathsf{B}(\ln(n)^A)$, and

$$J_0(t) \xrightarrow[t\to 0]{} 0$$

we necessarily have $\tau_n > \ln(n)^A$. **Can we do better?**

- Let $\mathcal{N} > 1;\ u_1, \dots, u_{\mathcal{N}} \in \mathbb{S}^1$ random and

$$R_{\mathcal{N}}(x) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \cos(\langle u_i, x\rangle).$$

- We want to show that for $\eta \in (0,1)$, for $\tau_n \sim \exp(\mathcal{N})$, whp

$$\sup_{x\in \mathsf{B}(\tau_n)} R_{\mathcal{N}}(x) < \eta$$

$\Rightarrow$ pseudo-periods are at least of scale $\exp(\mathcal{N})$.

Lower bound

Theorem (Dirichlet bound is almost optimal)

Assumptions:

- *The system* $(u_1, \dots, u_{\mathcal{N}})$ *is shift-invariant on* $\mathbb{S}^1$
- *The* $h(u_i)$ *satisfy the Hoeffding type inequality for* h *bounded smooth*

$$\mathbb{P}\left(\left| \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} h(u_i) - \mathbb{E}(h(u_1)) \right| > t \right) < \exp(-ct^{\gamma}\mathcal{N})$$

where $\gamma, c > 0$ *do not depend on* h.

Typical example: *i.i.d. uniform* u_i *on* $\mathbb{S}^1(\gamma = 2)$.

Then $R_{\mathcal{N}}$ *is* **not** *almost periodic at scale* $\exp(\mathcal{N}^{1-})$ *:*

$$\sup_{\|\tau\| \in [1, \exp(\mathcal{N}^{1-})]} R_{\mathcal{N}}(\tau) < \frac{1}{2}.$$

Surprising

Hence the proportion of $(u_1, \dots, u_{\mathcal{N}_n})$ such that $R_{\mathcal{N}_n}$ does not have a "Dirichlet" pseudo period $\tau_{\mathcal{N}_n} \sim \exp(\mathcal{N}_n^{1-})$ goes to 0.

- Either the $(u_1, \dots, u_{\mathcal{N}_n})$ such that $\|u_i\|^2 = n$ fall into this small subset of $(\mathbb{S}^1)^{\mathcal{N}_n}$ (i.e. the toy model of i.i.d. wavevectors u_i is not fit)
- Or there is full correlation between $\mathcal{N}_n^A$ and $\exp(\mathcal{N}_n^{1-\varepsilon})$ but no replication.

A more elaborate toy model

- Recall

$$n = \prod_{j=1}^{k} p_j^{\alpha_j} \prod_{i=1}^{l} q_i^{2\beta_i}$$

where $p_j = 2$ or $p_j \equiv 1$ and $q_i \equiv 3$. Furthermore, [Sartori 21] showed that for most $n \in \mathscr{S}$, $\forall j, \alpha_j = 1$.

- Recall that

$$p_j \equiv 1 \mod 4 \Leftrightarrow p_j = a_j^2 + b_j^2 = z_j \overline{z_j} \text{ with } z_j = a_j + ib_j$$

- Hence for most n, the $u = a + ib$ solutions of $|u|^2 = n$ are indexed by the $\eta = (\eta_j) \in \{-1, 1\}^k$ via

$$u_\eta := \prod_{j=1}^{k} (a_j + i\eta_j b_j) \times Z_n = \sqrt{n} \exp(i\theta_0) \prod_{j=1}^{k} \exp(i\eta_j \theta_j).$$

More elaborate toy model (Cont'd)

- The covariance function of the ARW is hence, with $k = \omega(n)$

$$r_n(t) = \frac{1}{\mathcal{N}_n} \sum_{\substack{\eta \in \{-1,1\}^{\omega(n)} \\ \nu \in \{\pm 1, \pm i\}}} \nu \cos\left(2\pi \frac{\langle u_\eta, t\rangle}{\sqrt{n}}\right)$$

$$= \frac{1}{\mathcal{N}_n} \sum_{\eta \in \{-1,1\}^{\omega(n)}, \nu} \nu \cos\left(2\pi \langle \exp(i\underbrace{\theta_0 + i \sum_j \eta_j \theta_j}_{\theta_\eta}), t\rangle\right)$$

- Consider the **Linearised** covariance function

$$s_n(t) = \frac{1}{\mathcal{N}_n} \sum_{\eta \in \{-1,1\}^{\omega(n)}} \cos(2\pi \theta_\eta |t|)$$

Important point: There are $\omega(n)$ degrees of freedom.

If the θ_η were iid, by Dirichlet Theorem, the smallest ε-period would be of the order roughly $\varepsilon^{-\mathcal{N}} \gg \exp(\ln(n)^{\ln(2)/2+})$.

Theorem

There is $1 \leqslant \|\tilde{\tau}_n\| \leqslant \ln(n)^{\ln(\ln(\ln(n)))}$ *such that*

$$s_n(\tilde{\tau}_n) \geqslant 1 - \exp(-\ln(n)^\delta).$$

We get closer to the scale $\ln(n)^A \sim \mathcal{N}_n^{A'}$.
Proof The θ_η are linear combinations of $\omega(n)$ many θ_j.
We modify The "Dirichlet principle lemma" to show that it is almost equivalent to the situation where $\mathcal{N} = \omega(n)$, with $2^{\omega(n)} = \mathcal{N}_n$. Then if $\ln(\varepsilon) \sim -\ln(n)^\delta$

$$\varepsilon^{-\omega(n)} = \varepsilon^{-\ln(\mathcal{N}_n)/\ln(2)} = \exp(-\ln(\varepsilon)\ln(\ln(2)\ln(\ln(n))/2\ln(2)))$$

Question: Does the ARW replicate at such scales?

Thank you for your attention!

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