Some Variations on the Mean Field Limit

Marta Leocata, joint work with F. Flandoli (SNS), M. Aleandri (SNS) and C. Ricci (Università di Pisa)

June 15, 2023

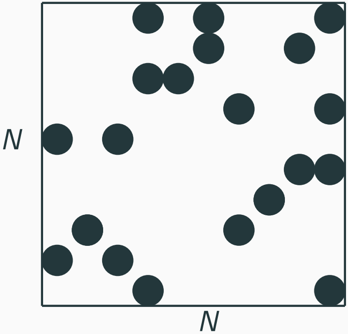
Summer School on Mean Field Model, Rennes

Plan of the Talk

- Mean Field Interaction vs Other Type of Interaction [joint work with F. Flandoli (SNS), C. Ricci (Univerisity of Pisa)]
  - a brief introduction
  - our contribution
- First Order Approximation [joint work with F. Flandoli (SNS), M. Aleandri(SNS)]
  - a brief introduction
  - technical difficulties

Mean Field Interaction vs Other Type of Interaction

An introduction to scaling limit



$$\left(X_t^{i,N}\right)_{i=1,\dots,K_{\bar\rho}} \in \left(\mathbb{T}_N^d\right)^{K_{\bar\rho}}$$

$$K_{\bar\rho} = \lfloor \bar\rho N^d \rfloor$$

$\bar\rho =$ density of particles

$\mathrm{supp} V =$ radius of a particle

$$dX_t^{i,N} = -\sum_{j=1}^{K_{\bar\rho}} \nabla V(X_t^{j,K} - X_t^{i,N})dt + \varepsilon dB_t^i \quad i = 1, \dots, K_{\bar\rho}$$

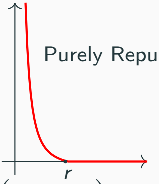
where $V : \mathbb{R}^d \to \mathbb{R}$ and B_t^i are independent Brownian motion in $\mathbb{R}^d$, $0 < \epsilon << 1$.

An introduction to scaling limit

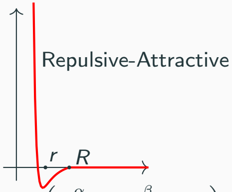
Training examples are related to biology, in particular to cellular adhesion. We consider V compactly supported and radial,

$$\mathrm{supp} V = B(0,1), \quad \nabla V(x) = V'(|x|)\frac{x}{|x|}.$$

Moreover we consider purely repulsive or attractive-repulsive potential,

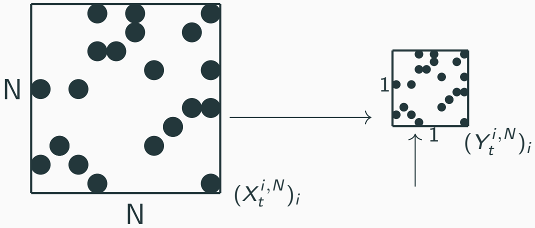


$V(|x|) = \left(\frac{1}{|x|^\alpha} - C\right) \mathbb{1}_{\{x \leq r\},\ \alpha > 0}$



$V(|x|) = \left(\frac{r^\alpha}{|x|^\alpha} - \frac{r^\beta}{|x|^\beta} + C\right) \mathbb{1}_{\{x \leq R\}\ \alpha > \beta}$

An introduction to scaling limit.



$$\left(Y_t^{i,N}\right)_{i=1,\ldots,K_{\bar\rho}} \in \left(\mathbb{T}_1^d\right)^K$$

$$K_{\bar\rho} = \lfloor \bar\rho N^d \rfloor, \rho \in (0,1),$$

• Scaling in space: in order to have $Y_t^{i,N} \in \mathbb{T}_1^d$, $Y_t^{i,N} = \frac{X_t^{i,N}}{N}$.

• Scaling in time: $Y_t^{i,N} = \frac{X_{t \cdot N^2}^{i,N}}{N}$, in order to see the brownian movement at the macro scale, we need to accelerate the time.

Note that $W_t = \frac{1}{\sqrt{c}} B_{ct}$ for each $c > 0$ is still a Brownian motion.

By Ito formula, calling $W^i_t = \frac{1}{N} B^i_{tN^2}$ and by some integration by substitution, we obtain the dynamic for $Y^{i,N}_t$:

$$dY^{i,N}_t = -N \sum_{j=1}^{K_{\bar\rho}} \nabla V(N\left(Y^{j,N}_t - Y^{i,N}_t\right))dt + \varepsilon dW^i_t \quad i = 1, \ldots, K_{\bar\rho}$$

$$dY^{i,N}_t = -\frac{1}{N^d} \sum_{j=1}^{K_{\bar\rho}} \nabla V_N(Y^{j,N}_t - Y^{i,N}_t)dt + \varepsilon dW^i_t \quad i = 1, \ldots, K_{\bar\rho}$$

$$\text{with } V_N(x) = N^d V(Nx)$$

then $\operatorname{supp} V_N = B(0, N^{-1})$.

An introduction to scaling limit

If we forget about the scaling and we assume $\nabla V_N(x) = \nabla V(x)$ we relapse on the Mean Field case.

$$\partial_t \rho = \frac{\epsilon^2}{2}\Delta\rho - \mathrm{div}((\nabla V * \rho)\rho)$$

- **Intermediate Interaction** $\nabla V_N(x) = N^{\beta d}\nabla V(N^\beta x)$, $\mathrm{supp} V_N = B(0, N^{-\beta})$ [Oeschlager '90]

$$\partial_t \rho = \frac{\epsilon^2}{2}\Delta\rho + \frac{C_V}{2}\Delta\rho^2$$

Volume fraction$\approx N^{(1-\beta)d}$

- **Local Interaction** $\nabla V_N(x) = N^d \nabla V(Nx)$, $\mathrm{supp} V_N = B(0, N^{-1})$ [Varadhan '91, Uchiyama '00]

$$\partial_t \rho = \Delta(P_V(\rho))$$

Volume fraction≈ 1

Macroscopic limit of Brownian particles with local interaction

Identity for the empirical measure

By Itô formula, if $\varphi : \mathbb{T}^d \to \mathbb{R}$ is a smooth compact support test function, then

$$d\varphi\left(Y_t^{i,N}\right) = -\left(\nabla\varphi\right)\left(Y_t^{i,N}\right)\frac{1}{N^d}\sum_{j\neq i}\nabla V_N\left(Y_t^{i,N} - Y_t^{j,N}\right)dt$$
$$+\left(\nabla\varphi\right)\left(Y_t^{i,N}\right)\epsilon dW_t^i + \frac{\epsilon^2}{2}\Delta\varphi\left(Y_t^{i,N}\right)dt.$$

Therefore, for $S_t^N := \frac{1}{N^d}\sum_{i=1}^{K_{\bar\rho}}\delta_{X_t^{i,N}}$,

$$d\left\langle\varphi, S_t^N\right\rangle = -\left\langle\nabla\varphi\int_{\mathbb{T}^d}\nabla V_N\left(\cdot - y\right)S_t^N\left(dy\right), S_t^N\right\rangle dt +$$
$$+\frac{\epsilon^2}{2}\left\langle\Delta\varphi, S_t^N\right\rangle dt + dM_t^{N,\varphi}$$

with $M_t^{N,\varphi} = \epsilon\frac{1}{N^d}\sum_{i=1}^{K_{\bar\rho}}\int_0^t\left(\nabla\varphi\right)\left(Y_t^{i,N}\right)dW_t$

Mean field case

If ∇V_N is independent of N, and it is continous and bounded, if $S^N \to \rho$

$$\lim_{N\to\infty} \left\langle \nabla\varphi \int_{\mathbb{T}^d} \nabla V\left(\cdot - y\right) S_t^N\left(dy\right), S_t^N \right\rangle =$$
$$= \lim_{N\to\infty} \left\langle \nabla\varphi(\nabla V * S_t^N), S_t^N \right\rangle = \langle \nabla\varphi(\nabla V * \rho), \rho \rangle$$

Then, passing to the limit in

$$d\left\langle \varphi, S_t^N \right\rangle = -\left\langle \nabla\varphi \int_{\mathbb{T}^d} \nabla V\left(\cdot - y\right) S_t^N\left(dy\right), S_t^N \right\rangle dt +$$
$$+ \frac{\epsilon^2}{2}\left\langle \Delta\varphi, S_t^N \right\rangle dt + dM_t^{N,\varphi}$$

we get the weak formulation of

$$\partial_t \rho = \frac{\epsilon^2}{2}\Delta\rho + \mathrm{div}((\nabla V * \rho)\rho)$$

Manipulation of the non linear term

$$\left\langle S_t^N, \nabla\varphi \cdot \nabla\left(V_N * S_t^N\right)\right\rangle = \frac{1}{N^{2d}} \sum_{i,j=1}^{K_{\bar\rho}} \nabla\varphi\left(Y_t^{i,N}\right) \cdot \nabla V_N\left(Y_t^{i,N} - Y_t^{j,N}\right).$$

By a symmetry argument, since $\nabla V_N(-x) = -\nabla V_N(x)$

$$= \frac{1}{N^{2d}} \sum_{i<j}^{K_{\bar\rho}} \left(\nabla\varphi\left(Y_t^{i,N}\right) - \nabla\varphi\left(Y_t^{j,N}\right)\right) \cdot \nabla V_N\left(Y_t^{i,N} - Y_t^{j,N}\right)$$

$$= \frac{1}{N^{2d}} \sum_{i<j} \sum_{\alpha=1}^{d} \partial_\alpha\left(\varphi\left(Y_t^{i,N}\right) - \varphi\left(Y_t^{j,N}\right)\right) \partial_\alpha V_N\left(Y_t^{i,N} - Y_t^{j,N}\right)$$

By Taylor expansion, $\psi_{\alpha,\beta} := \partial_\alpha V(x) \cdot (x)_\beta$ and $\psi := \partial_1 V(x) \cdot (x)_1$

$$\approx \frac{1}{N^{2d}} \sum_{i<j} \sum_{\alpha,\beta=1}^{d} \partial_\alpha\partial_\beta\varphi\left(Y_t^{j,N}\right)\left(Y_t^{i,N} - Y_t^{j,N}\right)_\beta \partial_\alpha V_N\left(Y_t^{i,N} - Y_t^{j,N}\right).$$

$$= \frac{1}{N^{2d}} \sum_{i<j} \sum_{\alpha,\beta=1}^{d} \partial_\alpha\partial_\beta\varphi\left(Y_t^{j,N}\right) N^d \psi_{\alpha,\beta}\left(N(Y_t^{i,N} - Y_t^{j,N})\right).$$

Local Equilibrium

Local equilibrium property: namely for each $\varphi, \psi \in C_c(\mathbb{T}^d)$. Let $\rho : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ and $\bar{\rho} = \int \rho(y)dy$ and $K = \lfloor \bar{\rho} N^d \rfloor$

$$\frac{1}{N^d} \sum_{i=1}^{K_{\bar{\rho}}} \varphi\left(Y_t^{j,N}\right) \sum_j \psi\left(N(Y_t^{i,N} - Y_t^{j,N})\right) =$$

$$= \frac{1}{N^d} \sum_j \varphi\left(Y_t^{j,N}\right) \sum_{|Y^i - Y^j| \leq \frac{r}{N}} \psi\left(N(Y_t^{i,N} - Y_t^{j,N})\right)$$

$$\to \int_{\mathbb{T}^d} \varphi(y) \Psi_V(\rho(y)) dy$$

Local Equilibrium

Given $\psi \in C_c(\mathbb{R}^d)$ (it is on of $\psi_{\alpha\beta}(x) = x_\beta \partial_\alpha V(x)$) $\overline{\rho} > 0$

$$\Psi_V(\bar{\rho}) := \lim_N \mathbb{E}_{\mu_G}\left[\frac{1}{N^d}\sum_{i,j=1}^{K_{\bar{\rho}}} \psi\left(x^i - x^j\right)\right], \qquad \textit{(Virial Formula)}$$

where μ_G, is the invariant measure of the SDEs system ,

$$\mu_G(d\mathbf{x}) = \frac{1}{Z_N}\exp\left(-\frac{1}{2}\sum_{i\neq j} V(x_i - x_j)\right) d\mathbf{x}$$

with $\mathbf{x} = (x_1, \ldots, x_K) \in (\mathbb{T}_N^d)^{K_{\bar{\rho}}}$. Interpretation:

$$\frac{1}{N^d}\sum_{i,j=1}^{K_{\bar{\rho}}} \psi\left(x^i - x^j\right) = \frac{1}{N^d}\sum_{i=1}^{K_{\bar{\rho}}} F\left(x_i\right), \text{ with } F\left(x_i\right) = \sum_{|x^i - x^j| \leq r} \psi\left(x^i - x^j\right)$$

Ψ_V is a sort of spatial average of local observables.

$$= \frac{1}{2N^d} \sum_{i,j=1}^{K_{\bar\rho}} \sum_{\alpha,\beta=1}^{d} \partial_\alpha \partial_\beta \varphi \left(Y_t^{j,N}\right) \psi_{\alpha,\beta}\left(N\left(Y_t^{i,N} - Y_t^{j,N}\right)\right)$$

$$= \frac{1}{2N^d} \sum_{\alpha,\beta=1}^{d} \sum_{i=1}^{K_{\bar\rho}} \partial_\alpha \partial_\beta \varphi \left(Y_t^{j,N}\right) \sum_{j:\left|Y_t^{i,N} - Y_t^{j,N}\right| \le \frac{r_1}{N}} \psi_{\alpha\beta}\left(N\left(Y_t^{i,N} - Y_t^{j,N}\right)\right).$$

$$\to \frac{1}{2} \int \partial_\alpha \partial_\beta \varphi(x)\, \Psi_V^{\alpha,\beta}(\rho_t(x)) dx.$$

Summing up, passing to the limit in the weak formulation

$$d\left\langle \varphi, S_t^N \right\rangle = -\left\langle \nabla\varphi \int_{\mathbb{T}^d} \nabla V_N(\cdot - y)\, S_t^N(dy), S_t^N \right\rangle dt + \\ + \frac{\epsilon^2}{2}\left\langle \Delta\varphi, S_t^N \right\rangle dt + dM_t^{N,\varphi}$$

we get the weak formulation of $\partial_t \rho = \frac{\epsilon^2}{2}\Delta\rho + \sum_{\alpha,\beta=1}^{d} \Delta\Psi_V^{\alpha,\beta}(\rho)$.

Manipulation of the non linear term

By an isotropy argument

$$\partial_t \rho = \frac{\epsilon^2}{2}\Delta\rho - \Delta\Psi_V(\rho),$$

whre $\Psi_V(\rho) = \Psi_V^{\alpha,\alpha}(\rho) = \Psi_V^{1,1}(\rho)$. We reformulate the PDE as

$$\partial_t \rho = \Delta P_V(\rho), \quad P_V(\rho) = \frac{\epsilon^2}{2}\rho - \Psi_V(\rho).$$

- [Varadhan, '91], for $d = 1$ and purely repulsive potential,
- [Uchyiama, '00], for repulsive potential and repulsive- attractive potential.
- [Flandoli, Leocata, Ricci, '20] investigations on the explicit form of $\Psi_V(\rho)$.

Investigation on Ψ_V

Our aim is to obtain some insights on Ψ_V

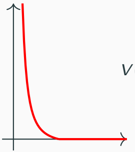
$$\Psi_V(\rho) = \lim_N \mathbb{E}_{\mu_G}\left[\frac{1}{N^d}\sum_{i,j=1}^{K_{\bar\rho}} \partial_1 V\left(x^i - x^j\right)\left(x^i - x^j\right)_1\right]$$

Two different tools:

- heuristic arguments based on intuitions on the Gibbs measure;
- numerics. By ergodicity property, we can produce realizations of such measure μ_G^ρ by simulations on large times of a SDE whose invariant measure is μ_G^ρ:

$$dX^i = -\sum_{j\neq i}^{K_{\bar\rho}} \nabla U(X_t^j - X_t^i)dt + \epsilon dB_t^i \tag{1}$$

Heuristical investigation on Ψ_V, Repulsive case, $d = 1$



$$V(|x|) = \left(\frac{1}{|x|^\alpha} - C\right) \mathbb{1}_{\{x \leq r\}},$$

Let us imagine realizations of points according to such measure. We look for an equilibrium configuration $(\overline{x}^{i,N})_{i=1,\dots,N}$ for a system of deterministic ODEs satisfying

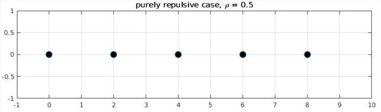
$$\dot{x}_t^{i,N} = -\sum_{j=1, j\neq i}^{K_{\bar{\rho}}} V'(x_t^{j,N} - x_t^{i,N}) \tag{2}$$

Then we approximate,

$$\mu_G \approx \frac{1}{N} \sum_{i=1}^{K_{\bar{\rho}}} \delta_{\overline{x}^i}, \quad \text{with } \overline{x}^i = \frac{i}{\bar{\rho}}, \; i = 1, \dots, K.$$

Heuristical investigation on Ψ_V, Repulsive case, $d = 1$

In the case of low density, $\bar{\rho} < 1/r$



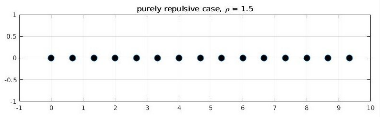
$$-\frac{1}{N^d}\sum_{i=1}^{K_{\bar{\rho}}}\sum_{j=1}^{K_{\bar{\rho}}} V'\left(\left|\overline{x}^{i,N}-\overline{x}^{j,N}\right|\right)\left|\overline{x}^{i,N}-\overline{x}^{j,N}\right| =$$

$$= -\frac{1}{N^d}\sum_{i=1}^{K_{\bar{\rho}}}\sum_{j=1}^{K_{\bar{\rho}}} V'\left(\left|\frac{i-j}{\bar{\rho}}\right|\right)\left|\frac{i-j}{\bar{\rho}}\right| =$$

$$= -\bar{\rho}\sum_{j=1}^{K_{\bar{\rho}}} V'\left(\frac{j}{\bar{\rho}}\right)\frac{j}{\bar{\rho}} = -\bar{\rho}\sum_{j=1}^{K_{\bar{\rho}}} \frac{\bar{\rho}^{\alpha+1}}{j^{\alpha+1}}\frac{j}{\bar{\rho}}\mathbb{1}_{\left|\frac{j}{\bar{\rho}}\right|\le r} = 0$$

Heuristical investigation on Ψ_V, Strong Repulsive case, $d = 1$

In the case of high density, $\bar{\rho} > 1/r$



$$\bar{\rho} \sum_{i=1}^{K} \frac{\bar{\rho}^{\alpha+1}}{i^{\alpha+1}} \frac{i}{\bar{\rho}} \mathbb{1}_{\left|\frac{i}{\bar{\rho}}\right| \le r} = \bar{\rho}^{1+\alpha} \sum_{i=1}^{\lfloor \bar{\rho} r \rfloor} \frac{1}{i^{\alpha}}$$

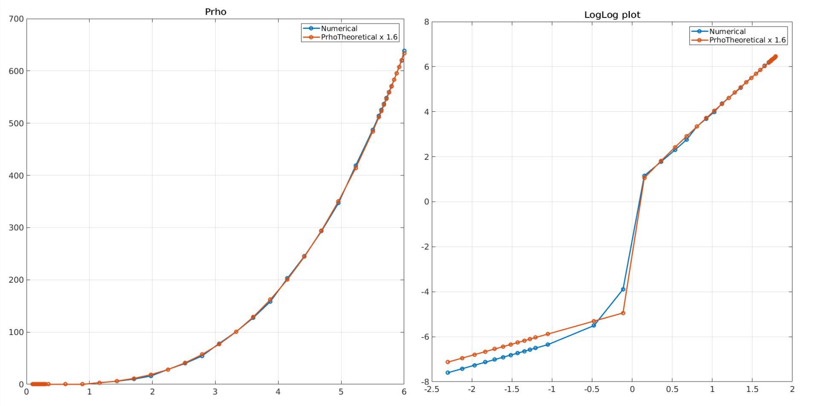
Since $\sum_{i=1}^{\bar{\rho} r} \frac{1}{i^\alpha} \approx C_\alpha^1 + \frac{C_\alpha^{\mathbf{2}}}{\bar{\rho}^{\alpha - \mathbf{1}}}$, we conclude

$$\bar{\rho}^{1+\alpha} \sum_{i=1}^{\lfloor \bar{\rho} R_{\mathbf{1}} \rfloor} \frac{1}{i^{\alpha}} = C_\alpha^1 \bar{\rho}^{\alpha+1} + C_\alpha^2 \bar{\rho}^2$$

Summing up

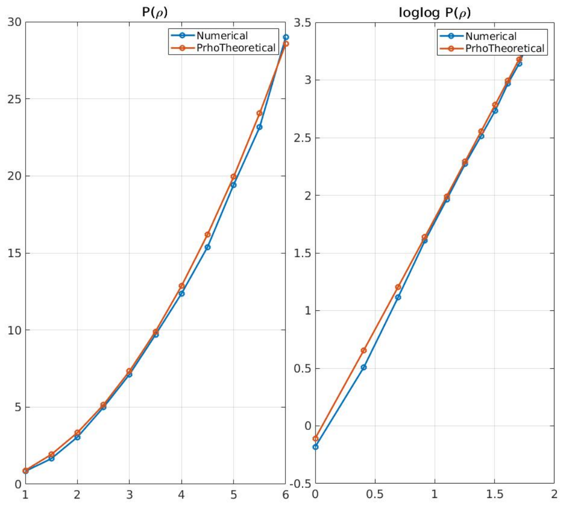
$$-\Psi_V(\bar{\rho}) = \begin{cases} 0 & \text{if } \bar{\rho} < r \\ C_\alpha^1 \bar{\rho}^{\alpha+1} + C_\alpha^2 \bar{\rho}^2 & \text{if } \bar{\rho} \ge r \end{cases}$$

Numerical investigation on Ψ_V, Strong Repulsive case, $d = 1$, $\alpha = 2$



If $f(x) = x^\alpha$, then $log(f(x)) = \alpha log(x)$. By a change of variable $y = log(x)$, $\log(f(x)) = \alpha y$. Then α will play the role of the angular coefficient in the loglog plot.

Numerical investigation on Ψ_V,Integrable potential, $d = 1$



Repulsive-Attractive case

By a similar argument, we approximate $\Psi_V(\rho)$ in the case of attractive-repulsive potential, $V(x) = \left(\frac{R^\alpha}{\alpha|x|^\alpha} - \frac{R^\beta}{\beta|x|^\beta} + C\right)\mathbb{1}_{|x|\leq r}$,

$$\Psi_V(\rho) = \begin{cases} 0 & \rho < r, \\ C_\alpha \rho^{1+\alpha} - C_\beta \rho^{1+\beta} - C_{\alpha,\beta}\rho^2 & \rho \geq r \end{cases}$$

First-order Approximation

Let's go back to the Mean Field case, where particles interact through a interaction kernel, K.

Motivating Example

Consider $\mu_0^N \to 0$, then the MF approximation does not hold. What is the equation to consider in this case?

By the Mean Field Limit:

$$\mu_t^N = \rho_t + \frac{1}{\sqrt{N}}.$$

By the Central Limit Theorem:

$$\mu_t^N = \rho_t + \frac{1}{\sqrt{N}}\eta_t + o\left(\frac{1}{\sqrt{N}}\right).$$

Our aim is to derive an equation furnishing a more detailed description of μ_t^N with respect to ρ_t.

Up to our knowledge (Suggestions are welcome)

- Diffusion Approximation in Ethier-Kurtz,
- Derivation of the first-order approximation in [Chevallier, Ost'20] for Hawkes Process.

Does $\rho_t^N := \rho_t + \frac{1}{\sqrt{N}}\eta_t$ satisfy an equation (at least at a formal level)?

$$
\begin{aligned}
d\rho^N(t) &= \Delta\rho(t) + \operatorname{div}\left((K * \rho(t))\rho(t)\right) + \frac{1}{\sqrt{N}}\Delta\eta(t) + \\
&\quad + \frac{1}{\sqrt{N}}\operatorname{div}\left((K * \eta(t))\rho(t)\right) + \frac{1}{\sqrt{N}}\operatorname{div}\left((K * \rho(t))\eta(t)\right) + dW_t[\rho_t] \\
&= \Delta\rho^N(t) + \operatorname{div}\left(F(\rho(t))\right) + \operatorname{div}\left(DF[\rho]\left(\frac{1}{\sqrt{N}}\eta(t)\right)\right) + dW_t[\rho_t]
\end{aligned}
$$

where W_t is a gaussian process on some negative sobolev Space ($W_0^{-(4+2D,D)}(\mathbb{R}^d)$, [Meleard '96]) with covariance

$$
\mathbb{E}\left[W_t(\varphi)W_s(\psi)\right] = \int_0^{t\wedge s} \rho_s \nabla\varphi \cdot \nabla\psi ds
$$

By Taylor expansion in infinite dimension,

$$F\left(\rho(t)+\frac{1}{\sqrt{N}}\eta(t)\right)=F(\rho(t))+DF[\rho]\left(\frac{1}{\sqrt{N}}\eta(t)\right)+ \\ +\frac{1}{2}D^2F[\rho](\frac{1}{\sqrt{N}}\eta(t),\frac{1}{\sqrt{N}}\eta(t))$$

Then,

$$\begin{aligned} d\rho^N(t) &= \Delta\rho^N(t)+\operatorname{div}(F(\rho(t)))+\operatorname{div}\left(DF[\rho]\left(\frac{1}{\sqrt{N}}\eta(t)\right)\right)+dW_t[\rho_t] \\ &= \Delta\rho^N(t)+\operatorname{div}\left(F(\rho^N(t))\right)+dW_t[\rho_t]-\frac{1}{2N}\operatorname{div}((K*\eta(t))\eta(t)) \end{aligned}$$

in other terms, ρ_t^N is almost a solution of a SPDE. "Almost" because there is an error term of order $o\left(\frac{1}{\sqrt{N}}\right)$

Our pourpose is to study the SPDE,

$$du^N(t) = \Delta u^N(t)dt + \text{div}\left(F(u^N(t))\right)dt + dW_t[\rho_t^N]$$

and to prove that v^N is in some sense close to u^N. Indeed, u^n

$$\begin{aligned}\mu_t^N &= u_t^N + (\rho_t + \frac{1}{\sqrt{N}}\eta_t - u_t^N) + o\left(\frac{1}{N}\right)\\ &= u_t^N + (\rho_t + \frac{1}{\sqrt{N}}\eta_t - u_t^N) + o\left(\frac{1}{N}\right) = u_t^N + o\left(\frac{1}{N}\right)\end{aligned}$$

u_t^N is close to μ_t^N at least as v_t^N, but differently from v_t^N, it satisfies an equation.

Difficulties:

- Is the Gaussian Process a Stochastic Integral in infinite dimension, $W_t = \text{div}\left(\sum_k \int_0^T \sqrt{v_t^N} e_k d\beta_t^k\right)$?
- If the previous is true, the process W_t will depend on $\sqrt{u_t^N}$. So $u_t^N \geq 0$ and in particular it must be a function.
- Well-posedness of W_t:

$$\mathbb{E}\left|\left|\int_0^t \nabla \cdot e^{(t-s)\Delta} \sum_k \sqrt{v_t^N} e_k d\beta_t^k\right|\right|_{L^2}^2 \leq$$

$$\leq \int_0^t \frac{c}{(t-s)} \sum_k \left|\left|\sqrt{v_t^N} e_k\right|\right|_{L^2}^2 dt = \infty$$

$\Rightarrow v_t^N$ should be a regular function.

-

$$\mathbb{E}\left|\left|v_t^N\right|\right|_{L^2} \leq \int_0^t \frac{c}{\sqrt{t-s}} \mathbb{E}\left|\left|v_s^N (K * v_s^N)\right|\right|_{L^2}^2 dt$$

- The natural space for v_t^N seems to be the space of fluctuations. But this space is not compatible with all the requirements above.

Thanks for the attention!