Mean field limits for systems of interacting neurons

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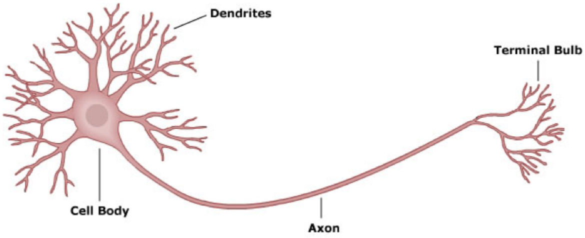
Outline

Where are we ?

Neurons



• Neurons : generate and propagate action potentials the long of their axons.

• They communicate by transmitting spikes : this is a fast transmembrane current of K^+/Na^+-ions, stimulated by ion pumps, and vehiculated by neurotransmitters (chemical substances).

Emission of spikes depends on integration of synaptic potentials and precise interplay with intrinsic properties of the cell. And on external stimuli and conditions. Picture shows the membrane potential of **one single neuron** in a potassium bath, under increasing concentration of potassium.

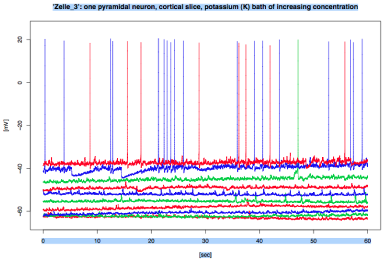


Figure: Cortical slice of an active network of $O(10^4)$ neurons, Picture by R. Höpfner and H. Luhmann, Mainz

Closer look to spikes

Superposition of all these spikes shows : The shape and the time duration of spikes is almost deterministic - and always “the same” (for a fixed neuron, under the same experimental conditions)

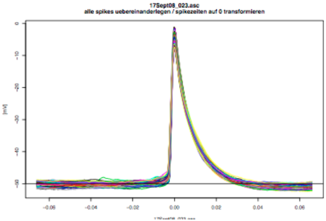
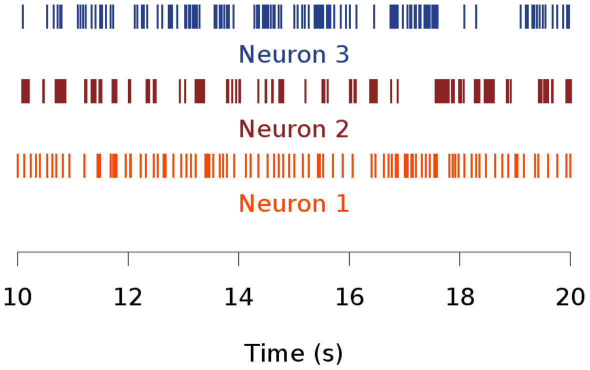


Figure: Picture by R. Höpfner, Mainz

The duration of each spike is very short (about 1 ms) - followed by a refractory period during which the neuron can not spike again (about 1 ms). So a description by means of *point processes* is reasonable.

Examples of spike trains

Use *spike sorting* (difficult and not evident) to obtain the *raster plot*, one way of representing *spike trains*:



Here, each neuron is represented by its successive spiking times.

So in the sequel we will represent systems of neurons by systems of interacting point processes where each point process represents the spiking times of a given neuron.

Where are we ?

Interacting neurons described by Hawkes processes

- N neurons that interact.
- Counting process associated to neuron $i \in \mathcal{I} = \{1, 2, \dots, N\}$: $Z^{N,i}(t) =$ number of spikes of neuron i during $[0, t]$.

Definition

Let $\mathcal{F}_t = \sigma(Z_s^{N,i}, s \leq t, i \leq N)$. Any $\mathcal{F}_t-$predictable positive process $\lambda^{N,i}(t)$ such that for all $0 \leq s \leq t$,

$$\mathbb{E}(Z_t^{N,i} - Z_s^{N,i} | \mathcal{F}_s) = \mathbb{E}[\int_s^t \lambda^{N,i}(u) du | \mathcal{F}_s]$$

is called (stochastic) intensity of $Z^{N,i}$.

– In other words,

$$P(Z^{N,i} \text{ has a jump during }]t, t + dt] | \mathcal{F}_t) = \lambda^{N,i}(t) dt.$$

– $\lambda^{N,i}(t)$ is the instantaneous jump rate of neuron i at time t.

The formula

$$P(Z^{N,i} \text{ has a jump during }]t, t+dt] | \mathcal{F}_t) = \lambda^{N,i}(t)dt$$

suggests a time discrete simulation scheme as follows:

- We bin time into small intervals of length δ.
- Within time $[n\delta, (n+1)\delta[$, we accept a spike of neuron i with probability
$$\lambda^{N,i}(n\delta)\delta.$$
There is no spike with probability
$$1 - \lambda^{N,i}(n\delta)\delta.$$
- As $\delta \to 0$, the waiting time up to the next time is described by a generalized exponentially distributed random variable, with stochastic and time dependent parameter $\lambda^{N,i}(t)$.

Hawkes intensity

- Intensity of $i-$th neuron given by

$$\lambda^{N,i}(t) = f_i\left(\sum_{j=1}^{N}\int_{[0,t[} h_{j\to i}(t-s)dZ^{N,j}(s)\right) = f_i\left(\sum_j \sum_{n: T_n^j<t} h_{j\to i}(t-T_n^j)\right),$$

where T_n^j all past spike times of neuron j.

- $f_i : \mathbb{R} \to \mathbb{R}_+$ non-decreasing, Lipschitz.
- $h_{j\to i} \in L^1_{loc}$ describes the influence of neuron j on neuron i.
- It also measures **how this influence vanishes as time goes by :** $h_{j\to i}(t-s)$ describes how a spike of neuron j lying back $t-s$ **time units in the past** influences the present spiking probability of neuron i.

The membrane potential

- The process

$$U^{N,i}(t) := \sum_{j=1}^{N} \int_{[0,t]} h_{j\to i}(t-s)dZ^{N,j}(s)$$

can be interpreted as membrane potential of neuron i at time t. It collects all the past spike events of its *presynaptic neurons.*

- It is a deterministic function of all past spiking times (which are random).
- The collection $(U^{N,i}(t))_{i\in\mathcal{I}}$ is not a Markov process (at least for general $h_{j\to i}-$kernels).
- Integrate-and-fire model : The neuron fires at a rate $f_i(U^{N,i}(t-))$ depending on the height of its actual membrane potential (just before the jump).

Example

$$h_{j\to i}(t-s) = W_{j\to i} e^{-\alpha_i (t-s)}.$$

$-$ $W_{j\to i}$ = synaptic weight of neuron j on neuron i. If $W_{j\to i} > 0$, then the synapse is **excitatory**, if $W_{j\to i} < 0$, then it is **inhibitory.**
$-$ Neurons which have a direct influence on i are those in

$$\mathcal{V}_i := \{j : W_{j\to i} \neq 0\} \quad \Rightarrow \textbf{Interaction graph}.$$

Where are we ?

- Like continuous processes adapted to the Brownian filtration which can be represented as stochastic integrals with respect to Brownian motion, point processes defined by means of their stochastic intensity can also be represented by means of some underlying *discrete noise*, which is a Poisson random measure.
- Poisson random measures are interesting objects per se - but we only use them for representation issues.

Let $X_n, n \geq 1$, all distinct, be random variables taking values in $\mathbb{R}_+ \times \mathbb{R}_+$. We put

$$\pi := \sum_{n \geq 1} \delta_{X_n}.$$

This is a random counting measure : $\pi(C) = \sum_{n=1}^{\infty} 1_C(X_n)$.

Definition

π is a Poisson random measure (PRM) on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity dtdx (Lebesgue measure) if
1) for all $C \in \mathcal{B}(\mathbb{R}_+^2)$,

$$\pi(C) \sim Poiss(|C|),$$

where $|C|$ denotes the Lebesgue measure of the set C,
2) for all $C_1, \ldots, C_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+)$ which are mutually disjoint, $\pi(C_1), \ldots, \pi(C_n)$ is an independent family of random variables.

Proposition (Which shows how to construct/simulate from $\pi_{|K}$ where $K \subset \mathbb{R}_+ \times \mathbb{R}_+$ is some compact set.)

Choose $N \sim$ *Poiss*$(|K|)$ *and, conditionally on* $N = n$, *choose* n *i.i.d. random variables* $X_1, \dots, X_n$ *which are uniformly distributed on* K *and independent of* N *(that is,* $P(X_1 \in C) = \frac{|C \cap K|}{|K|}$*). If we put*

$$\tilde{\pi} := \sum_{k=1}^{N} \delta_{X_k},$$

then $\tilde{\pi} \overset{\mathcal{L}}{=} \pi_{|K}$.

Basic facts on PRM's

- Let $\mathcal{F}_t^\pi := \sigma(\pi(A) : A \subset [0,t] \times \mathbb{R}_+, A \in \mathcal{B}(\mathbb{R}_+^2))$ and introduce the centered random measure $\tilde{\pi}(ds, dz) := \pi(ds, dz) - dsdz$.
- Then

$$M_t := \int_{[0,t]} \int_{\mathbb{R}_+} \varphi(s,z) \tilde{\pi}(ds, dz)$$

is a martingale for all predictable processes φ [1] s.t.

$$E \int_0^t \int_0^\infty |\varphi(s,z)| dsdz < \infty$$

for all $t > 0$. If moreover $\varphi \in L^2$, then

$$Var M_t = \int_0^t \int_0^\infty \varphi^2(s,z) dsdz.$$

[1] that is, $((s,\omega), z) \mapsto \varphi(s,z,\omega)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)-$measurable

Thinning

- Apply the above result with the particular choice

$$\varphi(s, z) = 1_{\{z \le \lambda(s)\}}.$$

- Then

$$Z_t = \int_{[0,t]} \int_{\mathbb{R}_+} 1_{\{z \le \lambda(s)\}} \pi(ds, dx)$$

  is a counting process having stochastic intensity λ.
- We say that Z is obtained from π by thinning. Kerstan 1964, Lewis+Shedler 1976, Ogata 1981.

Thinning/Poisson embedding

Theorem (Jacod 1979, Brémaud-Massoulié 1996)

Any (non-explosive simple) point process having stochastic intensity can be represented by means of the thinning of a PRM!

Coupling

Thinning helps for coupling:

- Suppose Z has intensity λ, and $\tilde{Z}$ intensity $\tilde{\lambda}$.
- Construct them according to the synchronous coupling, that is, using the same underlying PRM π (that makes them jump together as often as possible).
- Then uncommon jumps are caused by atoms (s, z) such that $\lambda(s) < z \leq \tilde{\lambda}(s)$ or $\tilde{\lambda}(s) < z \leq \lambda(s)$.
- So the total variation distance on $[0, t]$ is given by

$$\mathbb{E} \int_{[0,t]} |d(Z_s - \tilde{Z}_s)| = \mathbb{E} \int_0^t \mid \lambda(s) - \tilde{\lambda}(s)| ds.$$

Where are we ?

Mean field models

- We consider N similarly behaving neurons, that is, $f_i \equiv f$ Lipschitz and $h_{i \to j} = \frac{1}{N} h$, $h \in L^1_{loc}$ fixed kernel function.
- For any fixed N, we have a unique strong solution of the finite system, driven by N i.i.d. PRM's $\pi^i, 1 \le i \le N$.
- Each neuron jumps at rate $f(U^{N,i}(t-))$, where $U^{N,i}(t) =$

$$\sum_{j=1}^{N} \int_{[0,t]} h_{j \to i}(t-s) dZ^{N,j}(s) = \int_{[0,t]} h(t-s) d\bar{Z}^N(s) =: U^N(t),$$

$$\bar{Z}^N(t) = \frac{1}{N} \sum_{i=1}^{N} Z^{N,i}(t)$$

the empirical spike counting process.

- $Z^{N,i}(t) = \int_{[0,t] \times \mathbb{R}_+} 1_{\{z \le f(U^N(s-))\}} \pi^i(ds, dz)$: common intensity but independent driving noises.

Heuristics : MF Limit

- $U^N(t) = \int_{[0,t]} h(t-s) d\bar{Z}^N(s)$.
- As $N \to \infty$, we expect $\bar{Z}^N(t) \to \mathbb{E}(\bar{Z}(t))$, where $\bar{Z}(t)$ is the spike counting process of a typical neuron in an infinite limit population.
- $\bar{U}(t) = \int_{[0,t]} h(t-s) d\mathbb{E}\bar{Z}(s)$.

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- Thus
$$\bar{U}(t) = \int_{[0,t]} h(t-s) f(\bar{U}(s)) ds. \tag{1}$$

Theorem (Delattre, Fournier, Hoffmann 2016, Hawkes on large networks)

1) Under the condition $h \in L^1_{loc}$ *and* f *Lipschitz, there exists a unique solution of (1) such that* $t \mapsto \int_0^t f(\bar{U}(s))ds$ *is locally bounded.*

2) Consider the Sznitman coupling of $(Z^{N,i})$ *and* $(\bar{Z}^i)$: *for all* $1 \leq i \leq N$,

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Then we have the strong error bound

$$\mathbb{E}\left(\sup_{t \le T} |Z^{N,i}(t) - \bar{Z}^i(t)|\right) \le C_T N^{-1/2}.$$

– In particular, for any $l \geq 1$ fixed, we have convergence in law

$$(Z^{N,1}, \ldots, Z^{N,l}) \stackrel{\mathcal{L}}{\to} P^{\otimes l},$$

where $P = \mathcal{L}(\bar{Z})$ is the law of the McKean-Vlasov type non-linear limit process

$$\bar{Z}(t) = \int_{[0,t] \quad \times \mathbb{R}_+} 1_{\{z \leq d\mathbb{E}\bar{Z}(s)\}} \pi(ds, dz)$$

(we endow $D(\mathbb{R}_+, \mathbb{R})$ with the uniform convergence on compact time intervals).

– This means that we have propagation of chaos.

Elements of the proof.

Step 1.

f Lipschitz and h only locally integrable imply that the convolution equation

$$\bar{U}(t) = \int_{[0,t]} h(t-s) f(\bar{U}(s)) ds$$

possesses a unique solution in C^1.

Corollary

Let $\bar{U}(t)$ be the unique solution of (1) and let $\pi^i, i \geq 1$, be i.i.d. PRM's on $\mathbb{R}_+ \times \mathbb{R}_+$, having Lebesgue intensity. Then

$$\bar{Z}^i(t) := \int_{[0,t] \times \mathbb{R}_+} 1_{\{z \leq f(\bar{U}(s))\}} \pi^i(ds, dx)$$

is an i.i.d. family of time-inhomogeneous Poisson processes with intensity $f(\bar{U}(t))$, and

$$\mathbb{E}\bar{Z}^i(t) = \int_0^t f(\bar{U}(s)) ds.$$

Sznitman coupling

- $\Delta_t^{N,i} := \int_{[0,t]} |d(Z^{N,i}(s) - \bar{Z}^i(s))|$, $\delta_t^N = \mathbb{E}\Delta_t^{N,i}$ (does not depend on i by exchangeability).

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- $\sup_{s\leq t} |Z^{N,i}(s) - \bar{Z}^i(s)| \leq \Delta_t^{N,i}$.
- Then (supposing that f non-decreasing)

$$\Delta_t^{N,i} = \int_{[0,t]\times\mathbb{R}_+} 1_{\{f(U^N(s-)\wedge\bar{U}(s))<x\leq f(U^N(s-)\vee\bar{U}(s))\}} \pi^i(ds, dx).$$

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- Such that

$$\delta_t^N \le \mathbb{E}\int_0^t |f(\bar{U}(s)) - f(U^N(s))|ds$$

$$\le \|f\|_{Lip}\int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^N \int_{[0,s[} h(s-u)(d\mathbb{E}\bar{Z}(u) - dZ^{N,j}(u))\right| ds.$$

- To control

$$\int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^N \int_{[0,s[} h(s-u)(d\mathbb{E}\bar{Z}(u) - dZ^{N,j}(u))\right| ds$$

we add and subtract the limit process $d\bar{Z}^j(u)$ to obtain a decomposition $A+B$ where A is a variance and B a biais term.

- $$A = \int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^N \int_{[0,s[} h(s-u)(d\mathbb{E}\bar{Z}^j(u) - d\bar{Z}^j(u))\right| ds$$

- $$B = \int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1}^N \int_{[0,s[} h(s-u)(d\bar{Z}^j(u) - dZ^{N,j}(u))\right| ds.$$

Control of biais term by stochastic Fubini theorem

- Let $N(t)$ be any non-explosive simple counting process and $\varphi \geq 0$, then

$$\int_0^t \int_{[0,s[} \varphi(s-u)dN(u)ds = \int_0^t \varphi(s-u)N(u)du.$$

- Applying this to $N(u) \mapsto \Delta^{N,j}(u)$ and to $|h|$, and then taking expectation, we obtain

$$B \leq \int_0^t |h(t-u)|\delta^N(u)du.$$

Control of the variance term

- Put $X^{N,j}(t) = \int_{[0,t[} h(t-u) d\bar{Z}^j(u), 1 \leq j \leq N,$

Control of the variance term

- Put $X^{N,j}(t) = \int_{[0,t[} h(t-u) d\bar{Z}^j(u), 1 \leq j \leq N$, i.i.d. having mean $\int_0^t h(t-u) d\mathbb{E}(\bar{Z}^j(u))$.
- We rewrite

$$
\begin{aligned}
A = \int_0^t \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^N \int_{[0,s[} h(s-u) \left(d\mathbb{E}\bar{Z}^j(u) - d\bar{Z}^j(u) \right) \right| ds \\
= \int_0^t \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^N X^{N,j}(s) - \mathbb{E}(X^{N,j}(s)) \right| ds \\
\leq N^{-1/2} \int_0^t \sqrt{Var(X^{N,1}(s))} ds.
\end{aligned}
$$

- $$X^{N,1}(s) - \mathbb{E}X^{N,1}(s) = \int_{[0,s[} \int_{\mathbb{R}_+} h(s-u) 1_{\{x \le f(\bar{U}(u))\}} \tilde{\pi}^1(du, dx),$$

  where $\tilde{\pi}^1(du, dx) = \pi^1(du, dx) - dudx$ is the compensated PRM.

- So
$$Var(X^{N,1}(s)) = \int_0^s h^2(s-u) f(\bar{U}(u)) du.$$

- Since $h \in L^2_{loc}$ and $f(\bar{U}(u))$ a priori bounded on finite time intervals, this is upper bounded by C_T, for all $s \le t \le T$.

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- Since $h \in L^2_{loc}$ and $f(\bar{U}(u))$ a priori bounded on finite time intervals, this is upper bounded by C_T, for all $s \le t \le T$.
- Putting things together we obtain for all $t \le T$,

$$\delta_t^N \le C_T N^{-1/2} + C \int_0^t |h(t-s)| \delta_s^N ds.$$

Convolutional Gronwall

QUESTION : how to solve

$$\delta_t^N \leq C_T N^{-1/2} + C \int_0^t |h(t-s)| \delta_s^N ds. \tag{2}$$

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$$\delta_t^N \leq C_T N^{-1/2} + C \int_0^t |h(t-s)| \delta_s^N ds. \qquad (2)$$

Since $h \in L^1_{loc}$, there exists a sufficiently large A such that

$$\int_0^t |h(t-u)| 1_{\{|h(t-u) \geq A\}} du \leq \frac{1}{2C}.$$

So, since $\delta_s^N \leq \delta_t^N$,

$$\int_0^t |h(t-s)| \delta_s^N ds \leq \int_0^t A \delta_s^N ds + \frac{1}{2C} \delta_t^N.$$

Inserting in (2) and subtracting $\frac{1}{2}\delta_t^N$ on both sides implies, for all $t \leq T$,

$$\frac{1}{2}\delta_t^N \leq C_T N^{-1/2} + CA \int_0^t \delta_s^N ds.$$

Where are we ?

Erlang memory kernels

We now discuss how oscillations might arise in the limit model.

- Recall
$$\bar{U}(t) = \int_{[0,t]} h(t-s) f(\bar{U}(s)) ds.$$
- Consider *Erlang memory kernels*:
$h(t) = ce^{-\alpha t} \frac{t^n}{n!}, \alpha > 0, c \in \mathbb{R}, n \geq 0.$
- The delay of influence on the past is distributed.
- Takes its maximal value at n/α times steps back in the bast.
- We say that n is the *order of the memory*.

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- The delay of influence on the past is distributed.
- Takes its maximal value at n/α times steps back in the bast.
- We say that n is the *order of the memory*. If $c > 0$, the influence on the past is excitatory, else, inhibitory.
- Notice that $h'(t) = -\alpha h(t) + ce^{-\alpha t}\frac{t^{n-1}}{(n-1)!}$

Monotone cyclic feedback systems

- Introduce the *auxiliary variables*

$$x^k(t) = c\int_0^t e^{-\alpha(t-s)}\frac{(t-s)^{n-k}}{(n-k)!}f(\bar{U}(s))ds,\ 0 \le k \le n.$$

- So $\bar{U}(t) = x^0(t)$.

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- And

$$(+)\begin{cases} \frac{dx^k(t)}{dt} = -\alpha x^k(t) + x^{k+1}(t),\ 0 \le k < n \\ \frac{dx^n(t)}{dt} = -\alpha x^n(t) + f(x^0(t)) \end{cases}$$

Definition
System $(+)$ is called a monotone cyclic feedback system (Mallet-Paret+Smith 1990).

Some simulations in the case of a single neuron ($N = 1$)

A single neuron's spike train represented by a Hawkes process with an Erlang memory kernel, of memory order 3 :

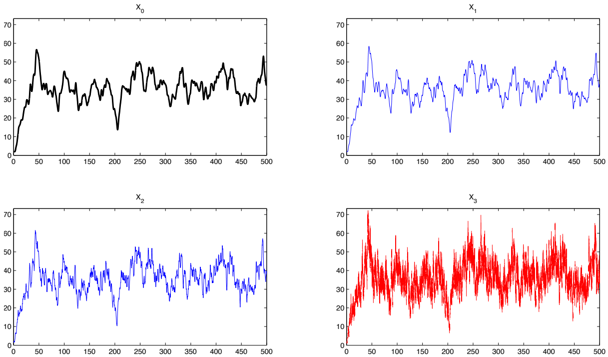


Figure: Picture by Aline Duarte, USP, Sao Paulo

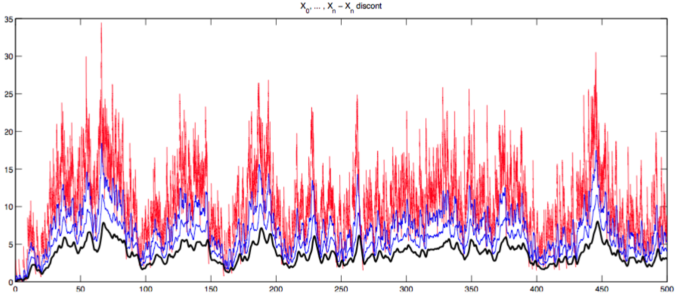


Figure: Picture by Aline Duarte, USP, Sao Paulo

Lemma

Suppose $c < 0$ and f non-decreasing. Then $(+)$ admits a unique equilibrium x^*.

Proof.

$x^* = (x^{*,0}, \ldots, x^{*,n})$ satisfies

$$\alpha x^{*,k} = x^{*,k+1} \text{ for all } \; k < n \text{ and } \alpha x^{*,n} = cf(x^{*,0}).$$

So

$$x^{*,n} = \frac{c}{\alpha} f(\frac{1}{\alpha^n} x^{*,n}).$$

Since $t \mapsto \frac{c}{\alpha} f(\frac{1}{\alpha^n} t)$ is non-increasing, this implies the existence of a unique solution. $\square$

Poincaré-Bendixson Theorem

Theorem (Mallet-Paret+Smith 1990)

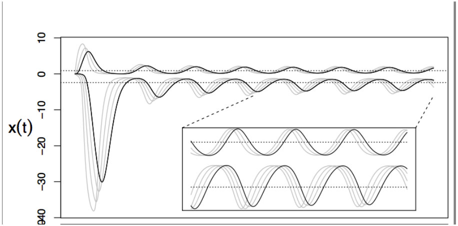
Suppose $n \geq 2$, f non-decreasing, bounded, analytic. Suppose moreover that

$$|cf'(x^{*,0})| > \frac{\alpha^{n+1}}{\left(\cos(\frac{\pi}{n+1})\right)^{n+1}}.$$

Then x^* is unstable and $(+)$ possesses a finite positive number of periodic orbits. At least one of them is orbitally asymptotically stable.

Oscillations of the limit intensities in a two-population model

Simulation of a system with 2 populations of neurons (we pass to a mean field limit in a multi-population frame) and memory 3 for the first population and memory 4 for the second one :



Where are we ?

Spatially structured Hawkes processes

- This part of the lecture is based on joint work with Julien Chevallier, Aline Duarte and Guilherme Ost.
- N neurons which are attached to positions $x_i \in \mathbb{R}^d, 1 \leq i \leq N$, x_i is the *position* of neuron i.
- Spiking rate of neuron i at time t is $f(U^{N,i}(t-))$ with

$$U^{N,i}(t) = e^{-\alpha t} u_0(x_i) + \frac{1}{N} \sum_{j=1}^{N} w(x_j, x_i) \int_{]0,t]} e^{-\alpha(t-s)} dZ^{N,j}(s).$$

- $w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the matrix of *synaptic weights*. (We read the interactions from left to right...)
- $\alpha \geq 0$ is the *leakage rate* and $u_0 : \mathbb{R}^d \to \mathbb{R}$ is the *initial input*.

Equivalent Markovian description

- Exponential leakage function $\Rightarrow$ Process of membrane potentials $(U^{N,i}(t), 1 \leq i \leq N)$ is a piecewise deterministic Markov process (PDMP).
- Generator : For $u = (u_1, \ldots, u_N)$

$$A^N g(u) = -\alpha \sum_{i=1}^{N} \frac{\partial g(u)}{\partial u_i} u_i + \\ + \sum_{i=1}^{N} f(u_i)[g(u + \frac{1}{N} \sum_j w(x_i, x_j) e_j) - g(u)],$$

e_j is the $j-$th unit vector in $\mathbb{R}^N$.

We shall work under

Assumption

1. *The firing rate f is Lipschitz.*
2. *The initial potential input $u_0(x)$ is bounded and Lipschitz in x.*
3. *The matrix of synaptic weights w is Lipschitz and bounded.*

Remark

What we actually need is this : Total input of interactions is bounded in the following sense: for some convenient a priori probability measure ϱ (spatial distribution of neurons)

$$\sup_x \int |w(y,x)|^2 \varrho(dy) < \infty$$

plus analogous condition when we integrate over x.

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$$\sup_x \int |w(y,x)|^2 \varrho(dy) < \infty$$

plus analogous condition when we integrate over x. The condition that w is **bounded** *is only to make the talk easier to follow.*

Proposition (Bounds on first and second moments)

Under our assumptions, for each $N \geq 1$ and $T > 0$:

$$\frac{1}{N}\sum_{i=1}^{N} E\left[(Z^{N,i}(T))\right] + \frac{1}{N}\sum_{i=1}^{N} E\left[(Z^{N,i}(T))^2\right] \leq C_T$$

where C_T does not depend on N.

For each $T > 0$, we consider the *empirical measure* of spike trains associated to positions of neurons

$$P^{(N,N)}_{[0,T]}(d\eta, dx) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\left((Z^{N,i}(t))_{0\leq t\leq T},\, x_i\right)}(d\eta, dx).$$

This is a *random* probability measure on $D([0,T], \mathbb{R}_+) \times \mathbb{R}^d$.

Let $\varrho(dx)$ be a probability measure on $\mathbb{R}^d$. We work under

Assumption

1. *The positions of neurons $x_1, \ldots, x_N$ are i.i.d. distributed according to $\varrho(dx)$.*
2. *ϱ possesses some exponential moments.*

Propagation of chaos

We will prove : there exists a *deterministic* probability measure $P_{[0,T]}$ on $D([0,T],\mathbb{R}_+) \times \mathbb{R}^d$ with :

$$\lim_{N\to\infty} d_{KR}(P^{(N,N)}_{[0,T]}, P_{[0,T]}) = 0, \ \textit{almost surely} \text{ w.r.t } x_1, x_2, \ldots.$$

Here:

$$d_{KR}(P^{(N,N)}_{[0,T]}, P_{[0,T]}) = \sup_{g\in Lip_1} \mathbb{E}\left[\left|\left\langle g, P^{(N,N)}_{[0,T]} - P_{[0,T]}\right\rangle\right|\right],$$

$Lip_1 = Lip_1(D([0,T],\mathbb{R}_+) \times \mathbb{R}^d)$ and $\mathbb{E}$ is taken w.r.t to the randomness present in the *jumps.*

Remark

d_{KR} is a Kantorovich-Rubinstein type distance.

Proposition (*A priori* properties of $P_{[0,T]}$:)

Under our assumptions,

$$\int_{\mathbb{R}^d} \int_{D([0,T],\mathbb{R}_+)} [\eta_T^2 + \eta_T] P_{[0,T]}(d\eta, dx) < \infty.$$

By the Disintegration theorem:

$$P_{[0,T]}(d\eta, dx) = P_{[0,T]}(d\eta | x) \varrho(dx).$$

MAIN QUESTION : what does $P_{[0,T]}(d\eta|x)$, the conditional distribution of η on $[0, T]$, given the position $x \in \mathbb{R}^d$, look like?

Macroscopic Model:

Recall : Intensity of neuron at position x_i, at time t given by

$$f\left(e^{-\alpha t}u_0(x_i) + \frac{1}{N}\sum_{j=1}^{N} w(x_j, x_i)\int_{]0,t]} e^{-\alpha(t-s)}dZ^{N,j}(s)\right).$$

$\Longrightarrow$ For each $x \in supp(\varrho)$: $P_{[0,T]}(d\eta|x) =$ law of *inhomogeneous Poisson process* $\bar{Z}^x(t)$ having intensity $(\lambda(t,x))_{0\le t\le T}$, where

$$\lambda(t,x) = f\Big(e^{-\alpha t}u_0(x) + \int_{\mathbb{R}^d} w(y,x)\int_0^t e^{-\alpha(t-s)}\lambda(s,y)ds\varrho(dy)\Big). \quad (3)$$

Rewriting the a priori bounds in terms of the intensities gives that it should be a priori true that

$$\int_{\mathbb{R}^d} \left[\left(\int_0^T \lambda(t,x) dt \right)^2 + \int_0^T \lambda(t,x) dt \right] \varrho(dx) < \infty \qquad (4)$$

$\Longrightarrow$

Proposition (Regularity)

Under our assumptions, for any solution λ *of the equation* (3) *such that* (4) *holds for all* $T > 0$*, we have:*

1. $\forall T > 0$, $\lambda \in C([0,T] \times \mathbb{R}^d, \mathbb{R}_+)$ *and* $\|\lambda\|_{[0,T]\times\mathbb{R}^d,\infty} < \infty$.
2. λ *is Lipschitz-continuous in the space variable, uniformly in time over compact time intervals.*

Proposition (Existence and uniqueness)

Define F from $C([0,T]\times\mathbb{R}^d,\mathbb{R}_+)$ to itself by

$$\lambda \mapsto F(\lambda)(t,x) = f\Big(e^{-\alpha t}u_0(x) + \int_{\mathbb{R}^d} w(y,x)\int_0^t e^{-\alpha(t-s)}\lambda(s,y)ds\varrho(dy)\Big).$$

Then for any $\lambda,\tilde{\lambda}\in C([0,T]\times\mathbb{R}^d,\mathbb{R}_+)$,

$$\|F(\lambda)-F(\tilde{\lambda})\|_{[0,T]\times\mathbb{R}^d,\infty} \le C(\alpha,L_f)(1-e^{-\alpha T}) \times \|\lambda-\tilde{\lambda}\|_{[0,T]\times\mathbb{R}^d,\infty}. \quad (5)$$

Remark

The inequality (5) + *a fixed point argument imply both existence and uniqueness of a solution of the equation* (3).

Theorem

Under our assumptions, almost surely (wrt the positions) we have that

$$d_{KR}\left(P^{(N,N)}_{[0,T]}, P_{[0,T]}\right) \le C_T\left(N^{-1/2} + W_2(\mu^{(N)}, \varrho)\right).$$

Corollary : Suppose $\varrho(dy) \ll \lambda(dy)$ with C^1-density and fix $x, \tilde{x} \in supp(\varrho)$. Consider "kernels" $\Phi_N(z), \tilde{\Phi}_N(z)$ s.t. as $N \to \infty$,

1. $\Phi_N(z)\varrho(dz) \xrightarrow{w} \delta_x(dz)$ and $\tilde{\Phi}_N(z)\varrho(dz) \xrightarrow{w} \delta_{\tilde{x}}(dz)$.
2. For $\varphi, \tilde{\varphi} : D([0,T], \mathbb{R}_+) \to [-1,1], \in Lip_1$, let

$$g_N(\eta, z) = \varphi(\eta)\Phi_N(z) \sim \varphi(\eta)\delta_x(z)$$

and

$$\tilde{g}_N(\eta, z) = \tilde{\varphi}(\eta)\tilde{\Phi}_N(z) \sim \tilde{\varphi}(\eta)\delta_{\tilde{x}}(z).$$

Then :

$$E\left[\left\langle g_N, P^{(N,N)}_{[0,T]}\right\rangle \left\langle \tilde{g}_N, P^{(N,N)}_{[0,T]}\right\rangle\right] \to \left\langle \varphi, P_{[0,T]}(\cdot|x)\right\rangle \left\langle \tilde{\varphi}, P_{[0,T]}(\cdot|\tilde{x})\right\rangle .$$

The activity near x is *asymptotically independent* of that near $\tilde{x}$.
Relating this result with multi-class propagation of chaos:

- $x = \tilde{x}$: chaoticity within a class.
- $x \neq \tilde{x}$: chaoticity between two different classes.

Remark (Neural field equation)

Write for each $t \geq 0$ and $x \in \mathbb{R}^d$

$$u(t,x) = e^{-\alpha t} u_0(x) + \int_{\mathbb{R}^d} w(y,x) \int_0^t e^{-\alpha(t-s)} \lambda(s,y) ds \varrho(dy).$$

Then

$$\lambda(t,x) = f(u(t,x)),$$

and $u(t,x)$ satisfies the neural field equation:

$$\frac{\partial u(t,x)}{\partial t} = -\alpha u(t,x) + \int_{\mathbb{R}^d} w(y,x) f(u(t,y)) \varrho(dy), \ u(0,x) = u_0(x).$$

Remark

Neural field equations *have been widely studied in the analytical and the neuro-scientifique literature, see e.g. Bressloff (2012).*

- *Standard choices for f : Sigmoid, piecewise linear function or a Heaviside function (to ease computations only!)*
- *Often : Symmetry assumption on w, e.g. Mexican hat $\Longrightarrow$ to generate oscillations.*
- *Choices of ϱ : Uniform over bounded set (Luçon and Stannat 2014), or Gaussian distribution (Bressloff 2012).*

Writing $U^{(N)}(t, x_i) := U^{N,i}(t)$, where

$$U^{N,i}(t) = e^{-\alpha t} u_0(x_i) + \frac{1}{N} \sum_{j=1}^{N} w\left(x_j, x_i\right) \int_{]0,t]} e^{-\alpha(t-s)} dZ_j^{(N)}(s)$$

is the potential of neuron in position x_i at time t,
we obtain the convergence of $U^{(N)}(t, x_i)$ to the solution $u(t, x)$.
Corollary 3: Under the conditions of Theorem 1, $\forall\ T > 0$ and almost all realizations of $x_1, x_2, \ldots$, it holds that

$$\lim_{N \to \infty} \mathbb{E} \left(\int_{\mathbb{R}^d} \int_0^T |U^{(N)}(t, x) - u(t, x)| dt \mu^{(N)}(dx) \right) = 0,$$

where $\mathbb{E}$ is taken w.r.t the randomness present in the jumps.

Sketch of proof of Theorem

Estimates on the distance $d_{KR}(P^{(N,N)}_{[0,T]}, P_{[0,T]})$ are done in 2 steps. The fundamental objects to study are:

$$
\begin{aligned}
P^{(N,N)}_{[0,T]}(d\eta, dx) &= \frac{1}{N}\sum_{i=1}^{N} \delta_{\left((Z^{N,i}(t))_{0\le t\le T}, x_i\right)}(d\eta, dx),\\
P^{(\infty,N)}_{[0,T]}(d\eta, dx) &= P_{[0,T]}(d\eta|x)\mu^{(N)}(dx),\\
P_{[0,T]}(d\eta, dx) &= P_{[0,T]}(d\eta|x)\varrho(dx).
\end{aligned}
$$

Step 1: To show that (discretisation in space and coupling à la Sznitman)

$$
d_{KR}\left(P^{(N,N)}_{[0,T]}, P^{(\infty,N)}_{[0,T]}\right) \le C_T[N^{-1/2} + W_2(\varrho, \mu^{(N)})].
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Basically the same proof as before.

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Basically the same proof as before.

Step 2: To show that

$$d_{KR}\left(P^{(\infty,N)}_{[0,T]}, P_{[0,T]}\right) \le C_T W_1(\varrho, \mu^{(N)}).$$

Proof of $d_{KR}(P^{(\infty,N)}_{[0,T]}, P_{[0,T]}) \leq C_T W_1(\varrho, \mu^{(N)})$

- Fix any coupling $W^{[N]}(dx, dy)$ of $\mu^{(N)}(dx)$ and $\varrho(dy)$.

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- Fix any coupling $W^{[N]}(dx, dy)$ of $\mu^{(N)}(dx)$ and $\varrho(dy)$.
- $\forall$ x, y, take the sync. coupling of $\bar{Z}^x$ and $\bar{Z}^y$. Let $g \in Lip_1$.
- $$\begin{aligned} &< g, P^{(\infty,N)}_{[0,T]} - P_{[0,T]} > \\ &\qquad = \mathbb{E}\left[\int g(\bar{Z}^x, x)\mu^{(N)}(dx) - \int g(\bar{Z}^y, y)\varrho(dy)\right] \\ &\qquad\qquad = \mathbb{E}\int_{\mathbb{R}^d \times \mathbb{R}^d} [g(\bar{Z}^x, x) - g(\bar{Z}^y, y)] W^{[N]}(dx, dy). \end{aligned}$$

Proof of $d_{KR}(P^{(\infty,N)}_{[0,T]}, P_{[0,T]}) \leq C_T W_1(\varrho, \mu^{(N)})$

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- $g \in Lip_1$ implies that $[\ldots] \leq \sup_{t \leq T} |\bar{Z}^x_t - \bar{Z}^y_t| + \|x - y\|$.

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- $g \in Lip_1$ implies that $[\ldots] \leq \sup_{t\leq T} |\bar{Z}^x_t - \bar{Z}^y_t| + \|x - y\|$.
- $\mathbb{E}\sup_{t\leq T} |\bar{Z}^x_t - \bar{Z}^y_t| \leq \int_0^T |\lambda(s, x) - \lambda(s, y)|\, ds \leq C_T\|x - y\|$, since λ Lipschitz in space, uniformly in time. $\square$

Some simulations: Propagation of activity

Simulations done by Julien Chevallier

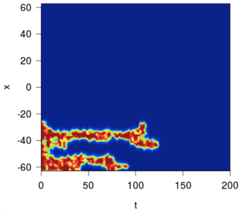

Figure: N=100

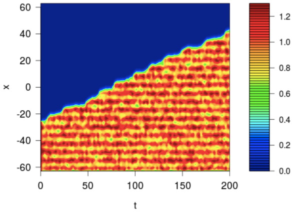

Figure: N=1000

Some simulations: Propagation of activity

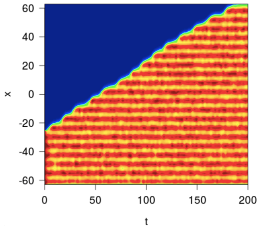

Figure: N=5000

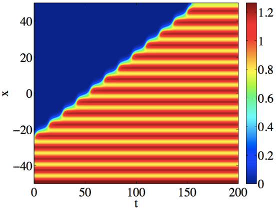

Figure: N=∞

Where are we ?

Models with reset

- Neurons reset after spiking (their potential goes back to a resting value, that we take equal to 0 here).
- N neurons having membrane potential $U^{N,i}(t) \geq 0, 1 \leq i \leq N$, each spiking at rate $f(U^{N,i}(t-))$.
- We take $h(t) = he^{-\alpha t}$, where $\alpha > 0$ (exponential decay), $h > 0$ (synaptic weight).

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- We take $h(t) = he^{-\alpha t}$, where $\alpha > 0$ (exponential decay), $h > 0$ (synaptic weight). Then

$$U^{N,i}(t) = \frac{h}{N} \sum_{j \neq i} \int_{]L_t^i, t]} e^{-\alpha(t-s)} dZ^{N,j}(s),$$

  $L_t^i = \sup\{s \leq t : \Delta Z^{N,i}(s) = 1\}$ last spike before time t.
- The membrane potential goes back to 0 at each spike: $t = L_t^i$ implies that $U^{N,i}(t) = 0$.

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  $L_t^i = \sup\{s \leq t : \Delta Z^{N,i}(s) = 1\}$ last spike before time t.
- The membrane potential goes back to 0 at each spike: $t = L_t^i$ implies that $U^{N,i}(t) = 0$. And then the neuron's potential collects inputs of presynaptic spikes since the last spiking time of the neuron.

- Equation driven by PRM's (of Lebesgue intensity)

$$dU^{N,i}(t) = -\alpha U^{N,i}(t)dt + \sum_{j\neq i} \frac{h}{N} \int_{\mathbb{R}_+} 1_{\{z \leq f(U^{N,j}(t-))\}} \pi^j(dt, dz)$$
$$- U^{N,i}(t-) \int_{\mathbb{R}_+} 1_{\{z \leq f(U^{N,i}(t-))\}} \pi^i(dt, dz).$$

- Generator: for all $\varphi : \mathbb{R}^N \to \mathbb{R}$, sufficiently regular, $x = (x_1, \ldots, x_N)$,

$$A^N \varphi(x) = \sum_{i=1}^N f(x_i)[\varphi(x + \Delta_i(x)) - \varphi(x)] - \alpha \sum_{i=1}^N \frac{\partial \varphi}{\partial x_i}(x) x_i,$$

$$(\Delta_i(x))_j = \begin{cases} h/N, i \neq j \\ -x_i, i = j. \end{cases}$$

- We call the reset the **big jump**.

Assumption

$f \uparrow$, $f(0) = 0$, $f(x) > 0$ for all $x > 0$, $f \in C^2$, convex,
$f(x + y) \le C_f(1 + f(x) + f(y))$ and

$$\sup_{x \ge 1}[f'(x)/f(x) + f''(x)/f'(x)] < \infty.$$

Example

We think of $f(x) = (x/K)^p$ for some (possibly large) p, where $K > 0$ is fixed (soft threshold).

Associated limit equation

- $$d\bar{U}^i(t) = -\alpha \bar{U}^i(t)dt - \bar{U}^i(t-)\int_{\mathbb{R}_+} 1_{\{z \le f(\bar{U}^i(t-)\}} \pi^i(dt, dz) + h\mathbb{E}(f(\bar{U}^i(t)))dt.$$
- Evolution remains stochastic as consequence of the "big jumps" (reset after spike).
- Writing $g_t = \mathcal{L}(\bar{U}^i(t))$, g_t is weak solution of a non-linear PDE

$$\partial_t g_t(x) = (\alpha x - hg_t(f))\partial_x g_t(x) + (\alpha - f(x))g_t(x), t \ge 0, x > 0,$$
$$g_t(f) = \int f(x) g_t(dx),\ g_t(0) = \frac{1}{h}\ \forall t > 0.$$

Theorem (with Nicolas Fournier, 2016)

1. *Suppose that* $g_0(f) < \infty$. *Then there exists a pathwise unique solution of the limit equation.*
2. *If moreover* $g_0(f^2) < \infty$, *for the Sznitman coupling, supposing that* $U^{N,i}(0)$ *i.i.d.* $\sim g_0$, *and introducing the function* $H(x) = f(x) + \arctan(x)$, *we have*

$$\sup_{t \le T} \mathbb{E}(|U^{N,i}(t) - \bar{U}^i(t)| + |H(U^{N,i}(t)) - H(\bar{U}^i(t))|) \le C_T/\sqrt{N}.$$

Plan of proof.

i) A priori bounds.

ii) Well-posedness of limit system (f unbounded and non-Lipschitz)

iii) Quantified propagation of chaos by Sznitman coupling.

A priori bounds

f non-decreasing implies that a priori

- $U^{N,i}(t) \leq U^{N,i}(0) + 3\bar{U}^N(0) + 4h\frac{1}{N}N_t^{Nf(2h)}$.

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- This implies the well-posedness of the finite system (we construct it up to time $\tau_K = \inf\{t : \sum_i U^{N,i}(t) \geq K\}$, it is well-defined up to τ_K, and then we show that $\lim_{K\to\infty} \tau_K = \infty$ thanks to the a priori bounds).

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- $\bar{U}^i(t) \leq \bar{U}^i(0) + 3\mathbb{E}(\bar{U}^i(0)) + 4hf(2h)t$.
- This implies that any solution of the limit equation a priori satisfies that $t \mapsto \mathbb{E}(f(\bar{U}(t)))$ is locally bounded, if $g_0(f) < \infty$.

Pathwise uniqueness of the limit

- Let $\bar{U}(t)$, $\tilde{U}(t)$ be two solutions, driven by the same PRM, starting from $\bar{U}(0) = \tilde{U}(0)$. Then

$$\begin{aligned}\bar{U}(t) - \tilde{U}(t) = -\alpha \int_0^t (\bar{U}(s) - \tilde{U}(s))ds \\ + h \int_0^t \mathbb{E}(f(\bar{U}(s)) - f(\tilde{U}(s)))ds \\ - \int_{[0,t] \times \mathbb{R}_+} \left(\bar{U}(s-) 1_{\{z \le f(\bar{U}(s-))\}} - \tilde{U}(s-) 1_{\{z \le f(\tilde{U}(s-))\}} \right) \pi(ds, dz).\end{aligned}$$

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which is not Lipschitz. So: compare the two processes after having applied the bijection $H(x) = f(x) + \arctan(x)$. Most important point, for $x > 0$ small, x is comparable to $\arctan(x)$ (while it is not to $f(x)$).

Properties of the space transform

Lemma

There is a constant C such that for all $x, y \in \mathbb{R}_+$, we have

1. $|H''(x)| \leq CH'(x)$,
2. $x + H'(x) \leq C(1 + f(x))$,
3. $|x - y| + |H'(x) - H'(y)| + |f(x) - f(y)| \leq C|H(x) - H(y)|$,
4. $-sign(x - y)(xH'(x) - yH'(y)) \leq C|H(x) - H(y)|$,
5. $$\begin{aligned} &-(f(x) \wedge f(y))|H(x) - H(y)| \\ &\qquad + |f(x) - f(y)|(H(x) \wedge H(y) - |H(x) - H(y)|) \\ &\qquad\qquad \leq C|H(x) - H(y)|. \end{aligned}$$

Proof.

Suppose $x \leq y$. Then we may rewrite the LHS of the last point as

$$-f(x)(H(y) - H(x)) + (f(y) - f(x))[H(x) - (H(y) - H(x))].$$

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We use that $(f(y)-f(x)) \le (H(y)-H(x))$ because $H(x)=f(x)+\arctan(x)$ with both f and arctan non-decreasing, that $f(y) \ge f(x)$ to get the upper-bound

$$|H(x)-H(y)|(H(x)-f(x)) = |H(x)-H(y)|\arctan x \le \frac{\pi}{2}|H(x)-H(y)|.$$

This completes the proof. $\square$

Pathwise uniqueness-continued

– Itô together with the above properties of H gives

$$\mathbb{E}(\mid H(\bar{U}(t)) - H(\tilde{U}(t))|) \leq C \int_0^t \mathbb{E}(\mid H(\bar{U}(s)) - H(\tilde{U}(s))|)ds + h \int_0^t B_s ds,$$

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$$B_s = \mathbb{E}\left(sign(\bar{U}_s - \tilde{U}_s)\left(H'(\bar{U}_s)\mathbb{E}f(\bar{U}_s) - H'(\tilde{U}_s)\mathbb{E}(f(\tilde{U}_s))\right)\right).$$

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$$B_s \leq \mathbb{E}(|H'(\bar{U}_s) - H'(\tilde{U}_s)|\mathbb{E}f(\bar{U}_s)ds + \mathbb{E}(H'(\tilde{U}_s))\mathbb{E}|f(\bar{U}_s) - f(\tilde{U}_s)|ds.$$

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Red terms bounded by a priori bounds since $H' \le C(1+f)$. Then use that

$|H'(x) - H'(y)| + |f(x) - f(y)| \le C|H(x) - H(y)|$. □

Existence of a strong solution of the limit equation

- OK if f is bounded, by Picard iteration.
- Less evident for f unbounded such as $(x/K)^p$.

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- OK if f is bounded, by Picard iteration.
- Less evident for f unbounded such as $(x/K)^p$.
- In this case we first prove the weak propagation of chaos.
- From this we deduce the existence of a weak solution of the limit equation.
- And thus strong existence (Yamada-Watanabe).

Theorem (with Nicolas Fournier, 2016)

Let $U^{N,i}(0)$ be i.i.d., $\sim g_0$ such that $g_0(f) < \infty$. Then

1. *The sequence of processes $(U^{N,1}(t))_{t\geq 0}$ is tight in $D(\mathbb{R}_+)$.*
2. *The sequence of empirical measures $\mu_N = N^{-1}\sum_{i=1}^N \delta_{(U^{N,i}(t))_{t\geq 0}}$ is tight in $\mathcal{P}(D(\mathbb{R}_+))$.*
3. *Any limit point μ of μ_N a.s. belongs to $\{\mathcal{L}((\bar{U}(t))_{t\geq 0})\}$.*
4. *Therefore, μ_N goes in probability to $\mu := \mathcal{L}((\bar{U}(t))$, where $\bar{U}$ is the unique solution to the limit equation.*

Remark

Point 2. follows from point 1., Sznitman, Saint Flour Lecture notes.

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Aldous criterion

Check that :

- for all $T > 0$, all $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \sup_{(S,S') \in A_{\delta,T}} \mathbb{P}(|U^{N,1}(S') - U^{N,1}(S)| > \varepsilon) = 0,$$

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  $A_{\delta,T}$ = set of all pairs of stopping times (S, S') such that $0 \leq S \leq S' \leq S + \delta \leq T$ a.s.,

- for all $T > 0$, $\lim_{K \uparrow \infty} \sup_N \mathbb{P}(\sup_{t \in [0,T]} U^{N,1}(t) \geq K) = 0$.

To check this in our model is not too difficult... (see details in paper with Nicolas).

Identification of the limit measure

- Once we have tightness of the sequence μ^N, we need to characterize any of its possible limits μ.
- **Martingale problem** We show that μ satisfies $F(\mu) = 0$, where for all $s_1 \leq s_2 \leq \ldots \leq s_k \leq s \leq t$,

$$F(\mu) := \int_{\mathcal{D}(\mathbb{R}_+)} \int_{\mathcal{D}(\mathbb{R}_+)} \mu(d\gamma)\mu(d\tilde{\gamma})\, \varphi_1(\gamma_{s_1}) \ldots \varphi_k(\gamma_{s_k}) \Big[\varphi(\gamma_t) - \varphi(\gamma_s) - \int_s^t f(\gamma_u)(\varphi(0) - \varphi(\gamma_u))du - \int_s^t \varphi'(\gamma_u)[hf(\tilde{\gamma}_u) - \alpha\gamma_u]du\Big] = 0.$$

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- This is nothing else then saying that μ has to be the law of $(\bar{U}(t))_t$.

Remark

– *To prove that any limit is solution of the martingale problem is done by* **developing the generator** *of the* $N-$*particle system (Taylor etc). Technical, but not so difficult.*

– *We can also prove a quantified version of the propagation of chaos by using a* C^2-*regularized version of the distance* $|H(x) - H(y)|$ *and some localization technique/truncation procedure of the total jump rate.*

Where are we ?

Longtime behavior of the finite system in the case with reset, without external stimuli

The results of this part are mostly based on a joint work with Pierre Monmarché.

- If $f(0) = 0$, the all-zero state is the only invariant state of the finite system.

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- If $f(0) = 0$, the all-zero state is the only invariant state of the finite system.
- Indeed, Aline Duarte and Guilherme Ost (2016) have shown :

Theorem

If f is differentiable in 0, then the system stops spiking almost surely. As a consequence, the unique invariant measure of the process is given by $\delta_{\mathbf{0}}$, where $\mathbf{0} \in \mathbb{R}^{\mathbf{N}}$ denotes the all-zero vector in $\mathbb{R}^N$.

Proof.

$-$ Suppose all initial potential values are $x_i > 0, 1 \leq i \leq N$, then the probability that the first spike of the system occurs after time t is

$$P(T_1 > t) = \exp\left(-\sum_{i=1}^{N}\int_0^t f(e^{-\alpha s}x_i)ds\right).$$

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– Let $t \to \infty$:

$$P(T_1 = \infty) = \exp\left(-\frac{1}{\alpha}\sum_{i=1}^{N}\int_0^{x_i}\frac{f(y)}{y}dy\right) > 0,$$

since $\int_0 \frac{f(y)}{y}dy < \infty$: $f'(0) < \infty$. □

Proof.

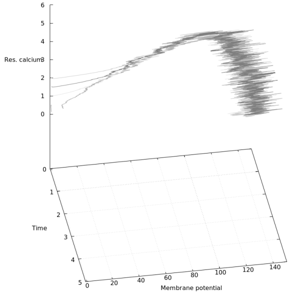
– At each time t such that **all potential values of all neurons** are simultaneously below some threshold K, there is a strictly positive probability

$$\geq \exp\left(-\frac{N}{\alpha}\int_0^K \frac{f(y)}{y}dy\right)$$

that none of the neurons does ever spike again.

– Use Lyapunov techniques to show that this event (all values below K) happens i.o. almost surely plus **conditional Borel-Cantelli lemma**. $\square$

- Finite system possesses a last spiking time $L = L^N < \infty$ almost surely, for any N.
- The situation changes however as $N \to \infty$ as we can see on simulations (done by C. Pouzat for a slightly different model)



- We will show that L^N is exponentially large in N as $N \to \infty$. But ... L^N is not a stopping time...

The last spiking time is a stopping time - somehow !

- L^N is not a stopping time of the process (that is, to decide if $L^N \leq t$ it is not sufficient to consider the history up to time t only).
- L^N becomes a stopping time if we consider a "larger" version $(U^N(t), E(t)) \in \mathbb{R}_+^{N+1}$ of our process.
- Fix an i.i.d. sequence $(\tau_n)_{n\geq 0}, \tau_n \sim Exp(1)$. Put $E(0) = \tau_0$.
- Up to the first jump time T_1, we have as before

$$dU^{N,i}(t) = -\alpha U^{N,i}(t)dt, 1 \leq i \leq N.$$

Moreover, we put

$$dE(t) = -\sum_{i=1}^N f(U^{N,i}(t))dt, \text{ that is,}$$

$$E(t) = E(0) - \int_0^t \sum_{i=1}^N f(U^{N,i}(s))ds.$$

- We define $T_1 = \inf\{t \geq 0 : E(t-) = 0\}$.
- At time T_1, the process U^N makes its transition as before (decide which of the neurons spike and then perform the jump transition).
- Moreover, we put $E(T_1) := \tau_1$, and start again with the dynamics described above up to the next jump

$$T_2 = \inf\{t \geq T_1 : E(t-) = 0\}.$$

- In this new setting,

$$L^N = \inf\{t : E(t) > \int_0^\infty \sum_{i=1}^N f(e^{-\alpha s} U^{N,i}(t)) ds\} \qquad (6)$$

  is now a stopping time with respect to the canonical filtration of the enlarged process (U^N, E).
- Notice that $\int_0^\infty \sum_{i=1}^N f(e^{-\alpha s} U^{N,i}(t)) ds$ is finite !!!
- A similar construction has been proposed by Marie Cottrell in her article *Mathematical analysis of a neural network with inhibitory coupling* SPA 1992.

Invariant states of the limit process

- In the limit, each neuron's potential undergoes leakage at exponential rate - and has an upward drift given by the current mean firing rate of the system (multiplied by h).
- Moreover, it spikes randomly, at rate $f(x)$, whenever its current value of potential is x.
- In any invariant state, the drift term $t \mapsto h\mathbb{E}(f(\bar{U}(t)))$ must be constant, say $\equiv b$.
- This defines - for any fixed b - a classical renewal Markov process $\bar{U}^b(t)$ (process coming back to 0 i.o. and thus being recurrent) with generator

$$A^b\varphi(x) = -\alpha x\varphi'(x) + b\varphi'(x) + f(x)[\varphi(0) - \varphi(x)]$$

and unique invariant probability measure π^b.

Shape of invariant measure for fixed b

- Kac Formula implies that for any $A \in \mathcal{B}(\mathbb{R}_+)$,

$$\pi^b(A) = \mathbb{E}_0 \int_0^{T^0} 1_A(\bar{U}^b(s))ds.$$

- Since in between successive jumps there is only the deterministic flow that acts, this equals

$$\int_0^\infty e^{-\int_0^t f(\varphi_s(0))ds} 1_A(\varphi_t(0))dt,$$

$\varphi_s(0) = \frac{b}{\alpha}(1 - e^{-\alpha s})$ solution of the deterministic flow.

- $\mathbb{R}_+ \ni s \mapsto \varphi_s(0) \in [0, b/\alpha[$ is a bijection. Use change of variables $x = \varphi_s(0)$ to obtain that π^b admits a Lebesgue density on $[0, b/\alpha[$ given by

$$g^b(x) = \frac{p^b}{b - \alpha x} e^{-\int_0^x \frac{f(y)}{b-\alpha y}dy}, \; x < b/\alpha,$$

where p^b is such that $\int_0^{b/\alpha} g^b(x)dx = 1$ [2].

[2] which implies that $\int f(x)g^b(x)dx = p^b$.

Remark

Any invariant measure of the true non-linear process must be solution of the fixed-point equation

$$hp^b = h\pi^b(f) = h\int f(x)g^b(x)dx = b.$$

- Since $f(0) = 0$, $b = 0$ and $\pi^0 = \delta_0$ is always a solution which corresponds to the silent state.
- Are there others?
- Is the all-zero state instable?

Non-trivial equilibrium states

Assumption

$f(0) = 0$, f bounded and Lipschitz, $f(u) \geq ku$ for all $u \in [0, u^*]$ for some $u^* > 0$.

Theorem

If $kh > \alpha$, then there exists at least a second equilibrium state π^{b^*} of the system with $b^* > 0$.

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In the sequel, just for simplicity, we assume that

$$f(x) = kx \wedge f_*, \text{ such that } f(x) = kx \text{ for all } \; x \leq f_*/k.$$

In particular, if $b \ll 1$, then $f(x) = kx$ on the support of π^b.

Proof.

- Fix $b > 0$ and consider the generator of the process

$$A^b\varphi(x) = (b - \alpha x)\varphi'(x) + f(x)[\varphi(0) - \varphi(x)] \ .$$

- $\varphi(x) = x$ gives $A^b\varphi(x) = b - \alpha x - xf(x)$.
- Integrating against the invariant measure gives

$$0 = \int [b - \alpha x - xf(x)]g^b(x)dx.$$

- Using that $f(x) \leq kx$ and that the support of π^b is $[0, b/\alpha]$, such that $x \leq b/\alpha$ on the support, we obtain

$$xf(x) \leq xk\, x$$

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$$xf(x) \leq xk\, x \leq x\, k\frac{b}{\alpha} \text{ such that}$$

$$0 \geq b - \left(\alpha + k\frac{b}{\alpha}\right) \int xg^b(x)dx.$$

$\square$

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- $p^b \leq \|f\|_\infty \Longrightarrow hp^b \leq h\|f\|_\infty < b$ for all b sufficiently large. Remains to show that $b \mapsto p^b$ continuous.

$\square$

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follows from

$$\frac{1}{p^b} = \int_0^{b/\alpha} \frac{1}{b - \alpha x} e^{-\int_0^x \frac{f(y)}{b-\alpha y} dy} dx$$
$$= \int_0^{1/\alpha} \frac{1}{1 - \alpha x} e^{-\int_0^x \frac{f(by)}{1-\alpha y} dy} dx,$$

(change of variables $x \mapsto x/b$, $y \mapsto y/b$). Dominated convergence.

Instability of 0 by means of an auxiliary simple Markov process

– Goal : Lower bound for the total spiking rate of the finite system.
– We introduce an auxiliary simple Markov process Z^N such that

$$F_t^N := \sum_{i=1}^{N} f(U^{N,i}(t)) \geq NZ^N(t)$$

for all t and such that large deviation estimates for Z^N are easily obtained (associated limit process z_t is deterministic).
– Construction of Z^N does only depend on behavior of derivative of f in vicinity of 0.
The true assumption that suffices is

Assumption

$f'(u)u \leq (r/\alpha)f(u)$, $f(u) \geq ku$ *for all* $u \leq u^*$.

– But in the sequel we suppose for simplicity that $f(x) = kx$ for all $x \leq f_*/k$. And that $kh > \|f\|_\infty$ and $N \gg 1$.

- If t is a spiking time, then all neurons with potential below $u^* := f^*/k - h/N$ will have an increase kh/N of their firing rate.

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- Therefore,

$$F_t^N \geq F_{t-}^N + \frac{kh}{N}\left(card\{i : U^{N,i}(t-) < u^*\}-1\right) - \|f\|_\infty.$$

- Since f is non-decreasing, $F_{t-}^N \geq f(u^*)card\{i : U^{N,i}(t-) \geq u^*\}$, such that

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$$\frac{F_t^N}{N} \geq \frac{F_{t-}^N}{N} + \frac{kh}{N}\left(1 - \frac{F_{t-}^N/N}{f(u^*)} - \frac{1}{N}\right) - \frac{\|f\|_\infty}{N} =: m_N\left(\frac{F_{t-}^N}{N}\right).$$

Definition of the PDMP Z^N

Z^N has generator

$$A^{Z^N}\varphi(z) = -rz\varphi'(z) + Nz[\varphi(m_N(z) \wedge z_N) - \varphi(z)],$$

where

$$m_N(z) = z + \frac{kh}{N}(1 - \frac{z}{f(u^*)} - \frac{1}{N})_+ - \frac{\|f\|_\infty}{N}; u^* = f^*/k - h/N,$$

$$z_N = \left(1 - \frac{\|f\|_\infty}{kh} - \frac{1}{N}\right)_+ f(u^*) - \|f\|_\infty/N.$$

Remark

Since $kh > \|f\|_\infty$, we have that $m_N(z) > 0$ for all $N \geq N_0$, and m_N is non-decreasing in z.

– The definition of z_N is such that

$$\begin{cases} F_{t-}^N/N \le z_N + \frac{\|f\|_\infty}{N} \Longrightarrow F_t^N \ge F_{t-}^N \\ F_{t-}^N/N \ge z_N + \frac{\|f\|_\infty}{N} \Longrightarrow F_t^N \ge z_N. \end{cases}$$

– So we may couple Z^N and F^N/N together s.t. they jump together as often as possible. Since $Z_t^N \le z_N$ for all t, the above implies that, whenever F^N jumps alone, after the jump we still have $F_t^N/N \ge Z_t^N$, provided it was true before the jump.

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– So we may couple Z^N and F^N/N together s.t. they jump together as often as possible. Since $Z^N_t \le z_N$ for all t, the above implies that, whenever F^N jumps alone, after the jump we still have $F^N_t/N \ge Z^N_t$, provided it was true before the jump.
– Since $m_N \uparrow$, at any common jump time t,

$$Z^N_t \le m_N(Z^N_{t-}) \le m_N(F^N_{t-}/N) \le F^N_t/N.$$

Mean field limit of Z^N

- Z^N jumps at rate Nz (whenever its current state is z), and jumps are of size $\sim \frac{kh}{N}(1-\frac{z}{f(u^*)})_+ - \frac{\|f\|_\infty}{N}$.
- In particular we have that, as $N \to \infty$,

$$A^{Z^N}\varphi(z) \to (-rx + xG(x))\varphi'(x)$$

which is the generator associated to a (non-stochastic) ODE given by

$$\dot{z} = -rz + G(z)z, \; G(z) = (kh(1-\frac{z}{f(u^*)}) - \|f\|_\infty)_+.$$

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- $G \downarrow$. Thus (if kh sufficiently large) there is a unique solution $z^* > 0$ of $G(z^*) = r$, and z^* is globally attracting. This implies the instability of 0 for the limit process and also a LDP.

The above coupling implies that, for

$$L^N = \inf\{t : \sum_{i=1}^{N} Z^{N,i}(]t, +\infty[) = 0\}$$

and

$$\bar{L}^N = \inf\{t : Z^N \text{ does not jump in }]t, \infty[\ \},$$

we have

Corollary

Suppose that $Z_0^N \leq \frac{F_0^N}{N} \wedge z_N$. *Then for the synchronous coupling,*

$$L^N \geq \bar{L}^N.$$

– It is however easier to control

$$L_\eta^N = \inf\{t : Z_t^N \leq \eta\}$$

for some fixed and small $0 < \eta < z^*$.

- If $Z_0^N = x \in]\eta, z^*[$, then $\lim_N Z_t^N = z_t \to z^*$ as $t \to \infty$.
- So for N large, Z_t^N should also be attracted to z^*; at least during some long time period.
- Such results can be expressed in terms of large deviation results - here for jump processes.

More precisely, we have

Theorem (Feng-Kurtz, LD for stoch. processes, 2006)

Suppose that $Z_0^N = x \in]\eta, z^*[$. Then there exists $\bar{V} < \infty$ (cost functional related to the fact that we force the limit ODE to go from its equilibrium z^* to η) such that

1. *For all $\delta > 0$,* $\lim_N \mathbb{P}_x(e^{(\bar{V}-\delta)N} < L_\eta^N < e^{(\bar{V}+\delta)N}) = 1$.
2. $\lim_N \frac{1}{N} \log \mathbb{E}_x L_\eta^N = \bar{V}$.

- In particular, the last spiking time of the system is exponentially large in N.
- We have even obtained a stronger result with Pierre Monmarché : the **metastability** of the system, that is, the fact that – suitably renormalized – exit times of neighborhoods of the limit invariant state are exponentially distributed.

Thanks for your attention !!!