How large is the mean-field framework? LLN and CLT results for diffusions on (random) graphs

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Outline

1. LLN for empirical measures for perturbations of mean-field diffusions
2. Fluctuation results

Diffusions interacting on a (random) graph

Let $\mathscr{G}_n$ a graph with vertices $V_n := \{1, \dots, n\}$ and for any $i, j \in V_n$, denote by $\varepsilon_{i,j}^{(n)} = 1$ (resp. $\varepsilon_{i,j}^{(n)} = 0$) if the edge $j \to i$ is present in $\mathscr{G}_n$ (resp. absent).
Consider n interacting diffusions $X_t^i \in \mathscr{X}$, $i = 1, \dots, n$, $t \geq 0$ solving

$$\mathrm{d}X_t^i = \frac{1}{np_n} \sum_{j=1}^{n} \varepsilon_{i,j}^{(n)} \Gamma(X_t^i, X_t^j) \mathrm{d}t + \mathrm{d}B_t^i, \; i = 1, \dots, n.$$

1. Also possible to add some local dynamics: $F(X_t^i)\mathrm{d}t$,
2. Additive noise: $B^1, \dots, B^n$: standard i.i.d. Brownian motions.
3. Assume $\mathscr{X} = \mathbb{T} := \mathbb{R}/2\pi$: each X^i is a phase on the torus (extensions to $\mathbb{R}^d$ possible).

We place ourselves in a situation where the $\varepsilon_{i,j}^n$ encode for a graph $\mathscr{G}_n$ that is well-approximated by a bounded graphon W (see the first example of P-E. Jabin's lecture).

$$\mathrm{d}X_t^i = \frac{1}{np_n}\sum_{j=1}^{n}\varepsilon_{i,j}^{(n)}\Gamma(X_t^i, X_t^j)\mathrm{d}t + \mathrm{d}B_t^i,\ i = 1,\dots,n.$$

Homogeneity

Further simplification here: assume that $W \equiv 1$: we are looking at homogeneous graphs.

The renormalisation by np_n is here to ensure that the interaction remains of order 1 as $n \to \infty$. If $d_n^i := \sum_{j=1}^n \varepsilon_{i,j}^{(n)}$ is the degree of vertex i, we will assume that $d_n \sim np_n$ as $n \to \infty$.

Remark: Extensions to inhomogeneous graphs/nontrivial bounded graphons possible and easy.

Dense vs Diluted

We are interested in two regimes

- The dense case $p_n \equiv p \in (0,1]$: the degree of each vertex remains of order n.
- The diluted case $p_n \to 0$ as $n \to \infty$: the degree of each vertex is $o(n)$.

A detour to the mean-field framework

In case $\mathscr{G}_n$ is the complete graph,

$$\mathrm{d}X_t^i = \frac{1}{n}\sum_{j=1}^{n}\Gamma(X_t^i, X_t^j)\mathrm{d}t + \mathrm{d}B_t^i,\ i = 1, \ldots, n.$$

Rewriting the interaction in terms of the empirical measure of the system

$$\mu_{n,t} := \frac{1}{n}\sum_{j=1}^{n}\delta_{X_t^j}$$

gives

$$\mathrm{d}X_t^i = \int_{\mathscr{X}}\Gamma(X_t^i, y)\mu_{n,t}(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t^i,\ i = 1, \ldots, n.$$

Crucial assumptions/properties are

- All-to-all interactions: the graph of interaction between particles in the complete graph K_n on $\{1, \ldots, n\}$,
- Homogeneous interactions: the strength of interaction is of the same order $\frac{1}{n}$, uniformly on all edges $(i \to j)$,
- Exchangeability at any time: assuming the initial condition $(X_0^1, \ldots, X_0^n)$ to be i.i.d., the law of the vector $(X_t^1, \ldots, X_t^n)$ at any $t > 0$ is invariant by permutation.

Theorem [McKean, Sznitman, etc.]

- The X^i, $i = 1, \dots, n$ have symmetric laws P_n on $\mathscr{C}([0,T),\mathscr{X})^n$ which are μ-chaotic, where μ is the law of the nonlinear process solution to
$$\begin{cases} \mathrm{d}\bar{X}_t &= \int \Gamma(\bar{X}_t, y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t, \\ \mu_t &= Law(\bar{X}_t). \end{cases}$$
- Equivalently, the empirical measure μ_n converges weakly as $n \to \infty$ towards μ, weak solution to the nonlinear Fokker Planck equation
$$\partial_t \mu_t = \frac{1}{2}\partial_x^2 \mu_t - \partial_x \left(\left\{ \int \Gamma(\cdot, y)\mu_t(\mathrm{d}y) \right\} \mu_t \right).$$
- The previous convergence may be formalized as
$$\mathbb{E}\left(\sup_{t\in[0,T]} d_{BL}(\mu_{n,t}, \mu_t) \right) \le \frac{C(T)}{\sqrt{n}}$$
for the bounded-Lipschitz distance defined as
$d_{BL}(\mu,\nu) = \sup\left\{ \left|\int f\mathrm{d}\mu - \int f\mathrm{d}\nu\right|, \|f\|_\infty \le 1, |f|_{Lip} \le 1 \right\}$.

From a modelling point of view, the homogeneity and exchangeability of the initial mean-field system

$$dX_t^i = \frac{1}{n}\sum_{j=1}^{n} \Gamma(X_t^i, X_t^j)dt + dB_t^i, \; i = 1, \ldots, n \tag{MF}$$

suffers from several limitations:

- in many applications, the graph of interaction is not complete (e.g. neuroscience) and interactions are not homogeneous along the graph,
- one may not want the initial condition to be i.i.d., only that (see [Gärtner, Oelschläger] for (MF))

  $$\mu_{n,0} \xrightarrow[n\to\infty]{\text{weakly}} \mu_0$$

  Just one motivation for this: in order to look at the behavior of (Markovian) (MF) on a time-scale that goes beyond bounded $[0, T]$, one may want to re-iterate the typical estimate

  $$\mathbb{E}\left(\sup_{t\in[0,T]} d_{BL}\left(\mu_{n,t}, \mu_t\right)\right) \leq \frac{C(T)}{\sqrt{n}}$$

  on $[T, 2T]$, $[2T, 3T]$, etc. [Bertini, Giacomin, Poquet, Coppini, L.]
  But at any T, the initial condition $\mu_{n,T}$ for $(\mu_{n,s}, s \in [T, 2T])$ is not i.i.d.!

Universality of the mean-field class

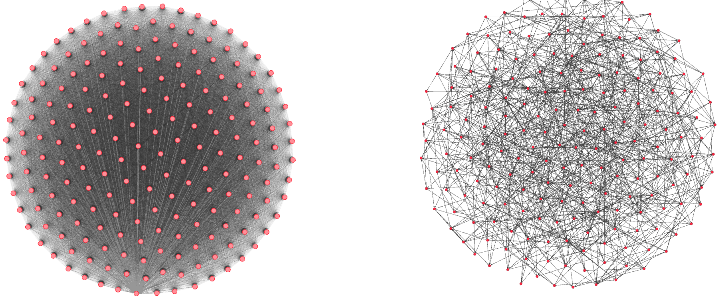
- How universal the mean-field framework is? How much can we perturb the complete graph of interaction K_n into some graph $\mathscr{G}_n$ and nonetheless conserve similar asymptotics (in particular the same mean-field limit) for the empirical measure as $n \to \infty$?
- At which level is this universality true? law of large numbers, fluctuations, large deviations?
- what is the possible range of dilution/sparsity of the graph $\mathscr{G}_n$?
- Is it possible to quantify the proximity of μ_n to its mean-field limit μ in terms of the proximity between $\mathscr{G}_n$ and K_n? for which graph topology?
- what does it imply on the local or global structure of the graph $\mathscr{G}_n$?

Example 1: Erdős-Rényi graph

$\mathscr{G}_n$: Erdős-Rényi graph with parameter $p_n \in [0,1]$: the $\varepsilon_{i,j}^{(n)}$ are independent variables with Bernoulli law with parameter p_n. Then d_n^i is Binomial$(n-1, p_n)$ so that $\mathbb{E}\left(d_n^i\right) \approx np_n$ so that we look at

$$\mathrm{d}X_t^i = \frac{1}{np_n} \sum_{j=1}^{n} \varepsilon_{i,j}^{(n)} \Gamma(X_t^i, X_t^j)\mathrm{d}t + \mathrm{d}B_t^i,\ i = 1, \ldots, n$$

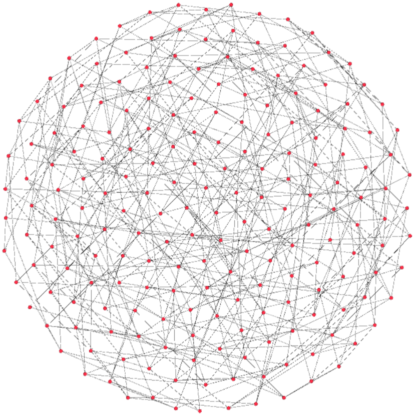
Complete graph VS Erdős-Rényi graph, with $n = 200$ and $p = 0.05$:



Example 2: Random regular graphs

One can construct a graph $\mathscr{G}_n$ in which each vertex has degree $d = d_n$ provided $3 \leq d < n$ and dn is even and defining $p_n = \frac{d_n}{n}$, so that one can also define

$$\mathrm{d}X_t^i = \frac{1}{np_n}\sum_{j=1}^{n} \varepsilon_{i,j}^{(n)} \Gamma(X_t^i, X_t^j)\mathrm{d}t + \mathrm{d}B_t^i, \; i = 1, \ldots, n$$



The LLN

$$\mathrm{d}X_t^i = \frac{1}{np_n}\sum_{j=1}^{n} \varepsilon_{i,j}^{(n)} \Gamma(X_t^i, X_t^j)\mathrm{d}t + \mathrm{d}B_t^i,\ i = 1, \ldots, n$$

Local empirical measures

The interaction is no longer a functional of the empirical measure μ_n but rather of a collection of local empirical measures

$$\mu_{n,t}^{(i)} := \frac{1}{np_n}\sum_{j=1}^{n} \varepsilon_{i,j}^{(n)} \delta_{X_t^j}.$$

Here $\mu_n^{(i)}$ accounts for the direct neighbbors of the vertex i. But the dynamics of a neighbor j of i itself depends on the local empirical measure $\mu_n^{(j)}$, etc. : a whole hierarchy of empirical measures appears, indexed by local patterns in the graph.

LLN: naive approach, synchronous coupling

Compare

1. the particle system

$$\mathrm{d}X_t^i = \frac{1}{np_n}\sum_{j=1}^{n} \varepsilon_{i,j}^{(n)} \Gamma(X_t^i, X_t^j)\mathrm{d}t + \mathrm{d}B_t^i,\ i = 1, \dots, n$$

2. with i.i.d. copies $\bar{X}_i$ of the nonlinear process, with same initial condition and Brownian motion

$$\mathrm{d}\bar{X}_t^i = \int \Gamma(\bar{X}_t^i, y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t^i,\ i = 1, \dots, n$$

As in the mean-field case, we want to prove that

$$\sup_{i=1,\dots,n} \mathbf{E}\left[\sup_{s\in[0,t]} \left|X_s^i - \bar{X}_s^i\right|^2\right] \xrightarrow[n\to\infty]{} 0.$$

Suppose that

$$b_n = b_n(\mathscr{G}_n) := \sup_{i=1,\dots,n} \left| \frac{d_n^i}{np_n} - 1 \right| \xrightarrow[n\to\infty]{} 0. \qquad (1)$$

Theorem [Delattre, Giacomin, L., 2016]

Suppose that X_0^i, $i = 1, \dots, n$ i.i.d. with law μ_0. Suppose that $np_n \to \infty$. Under Lipchitz regularity of Γ, assuming condition (1), there exists some constant $C_\Gamma > 0$ and n_0 such that for all $n \geq n_0$ and any $t \geq 0$,

$$\sup_{i=1,\dots,n} \mathbf{E}\left[\sup_{s\in[0,t]} \left|X_s^i - \bar{X}_s^i\right|^2\right] \leq C_\Gamma \left(\frac{1}{np_n} + b_n^2\right) \exp\left(C_\Gamma t\right) \xrightarrow[n\to\infty]{} 0. \qquad (2)$$

In particular,

$$\mathbf{E}\left[\sup_{s\in[0,t]} d_{BL}\left(\mu_{n,s}, \mu_s\right)\right] \xrightarrow[n\to\infty]{} 0, \qquad (3)$$

where $(\mu_t)_{t\in[0,T]}$ solves the NFP with initial condition μ_0.

The proof is elementary

Apply Ito's formula to $\left|X_t^i - \bar{X}_t^i\right|^2$:

$$\mathbf{E}\left[\sup_{0\leq s\leq t}\left|X_s^i - \bar{X}_s^i\right|^2\right] \leq C\int_0^t \mathbf{E}\left[\sup_{0\leq v\leq u}\left|X_v^i - \bar{X}_v^i\right|^2\right]\mathrm{d}u$$
$$+\int_0^t \mathbf{E}\left[\left|\frac{1}{np_n}\sum_{j=1}^n \varepsilon_{i,j}^{(n)}\Gamma\left(X_u^i, X_u^j\right) - \int \Gamma\left(\bar{X}_u^i, y\right)\mu_u(\mathrm{d}y)\right|^2\right]\mathrm{d}u$$

and split the last term within the integral into the sum of

- $\mathbf{E}\left[\left|\frac{1}{np_n}\sum_{j=1}^n \varepsilon_{i,j}^{(n)}\left\{\Gamma\left(X_u^i, X_u^j\right) - \Gamma\left(\bar{X}_u^i, \bar{X}_u^j\right)\right\}\right|^2\right]$, $\rightarrow$ Grönwall term
- $\mathbf{E}\left[\left|\frac{1}{np_n}\sum_{j=1}^n \varepsilon_{i,j}^{(n)}\Gamma\left\{\left(\bar{X}_u^i, \bar{X}_u^j\right) - \int \Gamma\left(\bar{X}_u^i, y\right)\mu_u(\mathrm{d}y)\right\}\right|^2\right]$, $\rightarrow$ covariance term, of order $\frac{1}{np_n}$
- $\left|\frac{1}{np_n}\sum_{j=1}^n \varepsilon_{i,j}^{(n)} - 1\right|^2 \mathbf{E}\left[\left|\int \Gamma\left(\bar{X}_u^i, y\right)\mu_u(\mathrm{d}y)\right|^2\right]$, $\rightarrow$ bounded by b_n^2.

About $b_n \to 0$ and the initial condition

Propagation of chaos

$$\sup_{i=1,\dots,n} \mathbf{E}\left[\sup_{s\in[0,t]} \left|X_s^i - \bar{X}_s^i\right|^2\right] \to 0 \tag{4}$$

is valid under the condition

$$b_n = b_n(\mathscr{G}_n) := \sup_{i=1,\dots,n} \left|\frac{d_n^i}{np_n} - 1\right| \xrightarrow[n\to\infty]{} 0.$$

- Example 1, ER(p_n): the last condition is satisfied for almost every realisation of ER(p_n) as long as

  $$np_n \gg \ln n$$

  This condition is optimal for (4), as $p_n \sim \frac{\ln n}{n}$ is the threshold for connectivity in ER(p_n) (but not necessarily optimal for the convergence of the empirical measure!)
- Example 2, regular graphs with degree d_n: last condition true when $d_n \to \infty$.

Why we are not very satisfied with this result

1. The condition $b_n \to 0$ does not even require the graph $\mathscr{G}_n$ to be connected (hence the result does not distinguish at all between (i) some $\mathrm{ER}(1/2)$ and (ii) two disjoints mean-field $K^{(1)}_{n/2} \cup K^{(2)}_{n/2}$. Solution to this "paradox": the convergence $\sup_{i=1,\dots,n} \mathbf{E}\left[\sup_{s\in[0,t]} \left|X^i_s - \bar{X}^i_s\right|^2\right] \to 0$ is only valid on small (logarithmic) times, but does not say anything on the long time behavior of the system.

Conclusion: in one needs to look at the longtime behavior of the system, $b_n \to 0$ is not the correct condition.

2. For the convergence of the empirical measure, for ER graphs, we need $np_n \gg \ln n$, but one expects the result to be valid under $np_n \to \infty$ only.
3. We want to discard the hypothesis that X^i_0 are i.i.d. One may even want that the initial condition depends on the graph !

Theorem, Convergence of μ_n, [Coppini, 2022, Coppini, L., Poquet, 2022]

Let $(\mathscr{G}_n)_n$ a (deterministic) sequence of graph on $\{1, \dots, n\}$. Suppose that $d_{BL}(\mu_{n,0}, \mu_0) \to 0$ (not necessarily i.i.d. and may depend on the graph). Then, if $\mathscr{G}_n$ satisfies

$$\|W_{\mathscr{G}_n} - \mathbf{1}\|_{\infty \to 1} = \sup_{s_i, t_j \in \pm 1} \frac{1}{n^2} \sum_{i,j=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) s_i t_j \xrightarrow[n\to\infty]{} 0 \qquad \text{(C)}$$

then

$$\mathbb{E}\left(\sup_{t\in[0,T]} d_{BL}(\mu_{n,t}, \mu_t) \right) \xrightarrow[n\to\infty]{} 0.$$

Examples

- Ex 1, ER(p_n) case: condition (C) is true as long as $np_n \to \infty$, and this condition is optimal: when $np_n \to \lambda$, $\mathscr{G}_n$ converges locally to a Galton-Watson tree [Oliveira, Reis, Stolerman 2020], [Lacker, Ramanan, Wu, 2020].
- Ex 2, regular graphs with degree d_n: condition (C) is true for Ramanujan graphs (i.e. regular graphs for which the second highest eigenvalue verifies $\lambda(d_n) \leq 2\sqrt{d_n - 1}$).

The effect on non-exchangeability

Consider Example 1: ER(p_n) graphs. What we have is: almost surely w.r.t. the graph,

1. Convergence of marginals: $\sup_{i=1,\ldots,n} \mathbf{E}\left[\sup_{s\in[0,t]} \left|X_s^i - \bar{X}_s^i\right|^2\right] \xrightarrow[n\to\infty]{} 0$,
   when $p_n \gg \frac{\ln n}{n}$ (and this is optimal!)
2. Convergence of empirical measure: $\mathbb{E}\left(\sup_{t\in[0,T]} d_{BL}\left(\mu_{n,t}, \mu_t\right)\right) \xrightarrow[n\to\infty]{} 0$,
   when $p_n \gg \frac{1}{n}$ (and this is optimal!)

Non-exchangeability breaks propagation of chaos

When $\frac{1}{n} \ll p_n \ll \frac{\ln n}{n}$, item (1) is false, but (2) is true: the convergence of marginals is no longer equivalent to the convergence of the empirical measure.

Recall that

$$\mu_{n,t}^{(i)} := \frac{1}{np_n} \sum_{j=1}^{n} \varepsilon_{i,j}^{(n)} \delta_{X_t^j}.$$

Theorem (Convergence of local measures, [Coppini, L., Poquet, 2022])

Let $\mathscr{G}_n$ be a sequence of ER(p_n) graphs. Suppose that $d_{BL}\left(\mu_{n,0}, \mu_0\right) \to 0$ (not necessarily i.i.d. but independent on the graph). Suppose that

$$p_n \gg \frac{1}{n^{1/3}},$$

then for almost every realisation of the graph $\mathscr{G}_n$, we have that, for all $l \geq 1$

$$\mathbb{E}\left(\sup_{t \in [0,T]} d_{BL}\left(\mu_{n,t}^{(l)}, \mu_t\right)\right) \xrightarrow[n \to \infty]{} 0$$

Sketch of proof

Define for all test function f and $s \leq T$, $P_{s,T}f(x) = \mathbb{E}_B\left(f(\Phi_s^T(x))\right)$ where $t \mapsto \Phi_s^t(x)$ solves $\mathrm{d}X_t = \int \Gamma(X_t, y)\,\mu_t(\mathrm{d}y) + \mathrm{d}B_t$ with $\Phi_s^s(x) = x$. Then, for any T, one has that $\partial_t \langle \mu_t\,,\, P_{t,T}\rangle = 0$. Hence, one expects that $\partial_t \langle \mu_{n,t}\,,\, P_{t,T}\rangle \approx 0$, up to order terms that one can control in n. More precisely, one has

$$\mathbf{E}\left|\langle \mu_T^n - \mu_T\,,\, f\rangle\right| \leq \mathbf{E}\left|\langle \mu_0^n - \mu_0\,,\, P_{0,T}f\rangle\right| + \mathbf{E}\left|\frac{1}{n}\sum_{k=1}^{n}\int_0^T \partial_x P_{t,T}f(X_t^k)\mathrm{d}B_t^k\right|$$
$$+ \int_0^T \mathbf{E}\left|\frac{1}{n^2}\sum_{i,j=1}^{n}\left(\frac{\varepsilon_{i,j}^n}{p_n} - 1\right)\partial_x P_{t,T}f(X_t^i)\Gamma\left(X_t^i, X_t^j\right)\right|\mathrm{d}t$$
$$+ \int_0^T \mathbf{E}\left|\frac{1}{n}\sum_{i=1}^{n}\partial_x P_{t,T}f(X_t^i)\left\langle \Gamma\left(X_t^i, \cdot\right)\,,\, \mu_t^n - \mu_t\right\rangle\right|\mathrm{d}t := (A) + (B) + (C) + (D)$$

- (A): initial condition
- (B): noise term
- (C): graph term, controlled by $\left\|W_{\mathscr{G}_n} - \mathbf{1}\right\|_{\infty\to 1}$, via Grothendieck inequality
- (D): controlled by $d_{BL}\left(\mu_{n,t}, \mu_t\right)$.

Extensions

1. Inhomogenous graphs:
   - [Medvedev, 2014]: no noise, Neunzert approach.
   - [L. 2020]: quenched results for diffusions on W-random graphs (bounded or in L^p)
   - [Bayraktar, Wu, 2020]: annealed results for bounded W
   - [Bet, Coppini, Nardi, 2020]: random graphons
2. Sparse connections:
   - [Lacker, Ramanan, Wu, 19-20], [Oliveira, Reis, Stolerman, 19-20]: interaction on locally tree-like graphs
   - [Jabin, Poyato, Soler, 2021]: sparse connection and measured-valued graphons
3. Beyond bounded times
   - [Coppini, 2022]: long-term stability for Kuramoto-type interaction
   - [Poquet, Le Bris, 2023]: uniform in time convergence for F confining
4. Other types of (jump) dynamics
   - [Agathe-Nerine, 2022]: Hawkes processes

Outline

Review of the pure mean-field case

Beyond the previous LLN result, one is interested in CLT results: the fluctuation process is

$$\eta_{n,t} := \sqrt{n}(\mu_{n,t} - \mu_t)$$

The process η_n is a signed measure, element of $\mathscr{C}([0,+\infty),\mathscr{S}')$, where $\mathscr{S}'$ is the classical Schwartz space of distributions.

Classical approaches in the pure mean-field case:

- Girsanov transform and asymptotics for U-statistics: finite-dimensional convergence of the field $\left\{\langle \eta_n, f\rangle, f \in L^2(\mu), \mathbb{E}_\mu(f) = 0\right\}$ towards some Gaussian process [Sznitman, Shiga, Tanaka, Hitsuda, Budhiraja, Wu, etc.]
- Semi-martingale approach [Fernandez, Méléard, Jourdain etc.]:
  1. Write a semimartingale decomposition for η_n,
  2. Prove tightness of η_n (typically in $\mathscr{C}([0,T],W^{-j,\alpha})$, where $W^{-j,\alpha}$ is the dual of the set of test functions g s.t. $\sum_{k\leq j}\int_{\mathscr{X}} \frac{\left|\partial_x^k g(x)\right|^2}{1+|x|^{2\alpha}}\mathrm{d}x < +\infty$, [Rebolledo,Mitoma, Joffe, Métivier]
  3. Identify the limit as the unique solution of a linear SPDE

The result of Fernandez and Méléard (1997) in the MF case

Ito's formula gives, for all test function f

$$\langle \eta_{n,t}\,,\, f\rangle = \langle \eta_{n,0}\,,\, f\rangle + \int_0^t \left\langle \eta_{n,s}\,,\, \mathscr{L}_{\mu_{n,s}}(f)\right\rangle \mathrm{d}s + W_{n,t}(f),$$

where

$$\mathscr{L}_\nu f := \frac{1}{2}\partial_x^2 f + \left\langle \nu(\mathrm{d}x'), \Gamma\left(\cdot, x'\right)\partial_x f(\cdot)\right\rangle + \left\langle \nu(\mathrm{d}x'), \Gamma\left(x', \cdot\right)\partial_x f(x')\right\rangle$$

and W_n is an explicit martingale converging to some Gaussian process W.

Theorem [Fernandez, Méléard, 1997]

There exist $j \geq 1$, $\alpha > 0$, such that for iid initial condition with sufficient moments, the process η_n converges in $\mathscr{C}([0,+\infty), W^{-j,\alpha})$ as $n \to \infty$ to η, unique solution to the linear SPDE

$$\eta_t = \eta_0 + \int_0^t \mathscr{L}^*_{\mu_s}\eta_s \mathrm{d}s + W_t,$$

Remark

- The proof relies heavily on the exchangeability of the initial condition
- The miracle of mean-field set-up again: we have a closed formulation of the fluctuation process.

Fluctuations on $\mathscr{G}_n = ER(p_n)$

There is no longer a closed equation for the fluctuation process: Ito's formula gives again

$$\eta_{n,t} = \eta_{n,0} + \int_0^t \mathscr{L}^*_{\mu_{n,s}} \eta_{n,s} \mathrm{d}s + \int_0^t \Theta^* \hat{\eta}_{n,s} \mathrm{d}s + W_{n,t},$$

for the auxiliary process:

$$\hat{\eta}_{n,t} = \frac{1}{n^{3/2}} \sum_{i,j=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) \delta_{(X_t^i, X_t^j)}$$

Question

If ones believes in the universality of the mean-field fluctuations, one is left with proving that

$$\hat{\eta}_n \to 0, \text{ in some } H^{-j}(\mathbb{T}) \text{ as } n \to \infty$$

Dealing with the auxiliary process: first wrong approach

$$\hat{\eta}_{n,t} = \frac{1}{n^{3/2}} \sum_{i,j=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) \delta_{(X_t^i, X_t^j)}$$

First guess: apply $\hat{\eta}_n$ to the test function $f \equiv 1$:

$$\langle \hat{\eta}_n , 1 \rangle = \frac{1}{n^{3/2}} \sum_{i,j=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right)$$

By Bernstein inequality, one has immediately that for all $\varepsilon > 0$

$$\mathbb{P}\left(|\langle \hat{\eta}_n , 1 \rangle| > \frac{1}{n^{\frac{1}{2}-\varepsilon} p_n^{1-\varepsilon}} \right) \leq 2 \exp\left(-\frac{n^{\varepsilon}}{4} \right)$$

and hence $\langle \hat{\eta}_n , 1 \rangle \to 0$ a.s. when $p_n \gg \frac{1}{n^{1/2-\varepsilon}}$.

Problem

The presence of $\delta_{(X_t^i, X_t^j)}$ makes the previous argument no longer applicable. X_t^i, X_t^j depend in a nontrivial way on the sequence $(\varepsilon_{i,j}^n)$: independence is broken.

$$\hat{\eta}_{n,t} = \frac{1}{n^{3/2}} \sum_{i,j=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) \delta_{(X_t^i, X_t^j)}$$

Idea: apply again Ito's formula and pursue this decomposition on $\hat{\eta}_n$:

$$\hat{\eta}_{n,t} = \hat{\eta}_{n,0} + \int_0^t \hat{\mathscr{L}}^*_{\mu_{n,s}} \hat{\eta}_n, s\mathrm{d}s + C_{n,t} + \hat{W}_{n,t},$$

Again, the remaining term C_n depends itself on higher statistics within the graph, i.e. e.g. quantities such that

$$\frac{1}{n^3} \sum_{i,j,k=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) \left(\frac{\varepsilon_{i,k}^{(n)}}{p_n} - 1 \right) \delta_{(X_t^i, X_t^j, X_t^k)}$$

etc.

Question

How to close this hierarchy of empirical measure?

Main result

Good news: the remainder term C_n vanishes as $n \to \infty$: the fluctuations are only captured by the 2-order expansion $(\eta_n, \hat{\eta}_n)$.

Theorem [Coppini, L., Poquet, 2022]

There exists $3 < r < r'$ such that if the initial fluctuations $(\eta_{n,0}, \hat{\eta}_{n,0})$ satisfies $\sup_n \mathbf{E}\left(\|\eta_{n,0}\|_{-r}^{1+\alpha}\right) < \infty$, $\sup_n \mathbf{E}\left(\|\hat{\eta}_{n,0}\|_{-r}^{1+\alpha}\right) < \infty$ and converge in $H^{-r'}(\mathbb{T}) \otimes H^{-r'}(\mathbb{T}^2)$ and and if the dilution condition holds

$$p_n \gg \frac{1}{n^{1/4}}.$$

Then, for almost every realisation of the graph $\mathscr{G}_n$, $(\eta_n, \hat{\eta}_n)$ converges in $\mathscr{C}\left([0,T], H^{-r'}(\mathbb{T}) \otimes H^{-r'}(\mathbb{T}^2)\right)$ towards the unique solution to the system of coupled SPDEs

$$\begin{cases} \eta_t = \eta_0 + \int_0^t \mathscr{L}_{\mu_s}^* \eta_s \mathrm{d}s + \int_0^t \Theta^* \hat{\eta}_s \mathrm{d}s + W_t, \\ \hat{\eta}_t = \hat{\eta}_0 + \int_0^t \hat{\mathscr{L}}_{\mu_s}^* \hat{\eta}_s \mathrm{d}s, \end{cases}$$

where $(W_t)_{t \in [0,T]}$ is an explicit Gaussian process, independent of $(\eta_0, \hat{\eta}_0)$.

Universality of fluctuations

Corollary

Suppose in addition that the initial condition is chosen independently of the graph. Then, under the previous assumption, the fluctuations are the same as in the mean-field case: $\hat{\eta} \equiv 0$ and η_n converges to

$$\eta_t = \eta_0 + \int_0^t \mathscr{L}^*_{\mu_s} \eta_s \mathrm{d}s + W_t$$

Hence, the mean-field CLT is universal, provided we choose the initial condition independently on the graph.

Remark

It is possible to choose well-prepared initial conditions (that depend on the graph) such that the fluctuations are captured by $(\eta, \hat{\eta})$, not only η. Hence, fluctuations are non universal is one chooses initial conditions that depend on the graph.

CLT for local fluctuations

We are also interested in the joint convergence of

$$\left(\zeta_n^1, \zeta_n^2, \eta_n\right)$$

where ζ_n^l is the fluctuation field associated to the local empirical measure

$$\zeta_n^l := \sqrt{np_n}\left(\mu_n^l - \mu\right).$$

Theorem [Coppini, L., Poquet, 2022]

Suppose that $(X_0^1, \dots, X_0^n)$ are i.i.d. with law μ_0, independent from the graph. Suppose that $\liminf_n np_n^5 = \infty$ and denote by $p := \lim_{n\to\infty} p_n \in [0,1]$. Then, for a.e. realizations of the graph, $\left(\zeta^{n,1}, \zeta^{n,2}, \eta^n\right)$ converges as $n \to \infty$ in $\mathscr{C}\left([0,T], (\mathscr{S}')^3\right)$ to $\left(\zeta^1, \zeta^2, \eta\right)$ solution to

$$\begin{cases} \zeta_t^l = \zeta_0^l + \int_0^t \mathscr{U}_s^* \zeta_s^l \mathrm{d}s + \sqrt{p} \int_0^t \mathscr{V}_s^* \eta_s \mathrm{d}s + W_t^l, \; l = 1, 2, \\ \eta_t = \eta_0 + \int_0^t \mathscr{L}_{\mu_s}^* \eta_s \mathrm{d}s + W_t. \end{cases} \tag{5}$$

for explicit linear operators $\mathscr{U}_s$, $\mathscr{V}_s$ and for $(\zeta_0^1, \zeta_0^2, \eta_0) \perp\!\!\!\perp \left(W_t^1, W_t^2, W_t\right)$ Gaussian processes with explicit covariance.

Consequence: phase transitions for local measures

Corollary

Under the previous assumptions,

- if $p_n \to p > 0$ (dense case), $(\zeta^1, \zeta^2) = \lim_{n\to\infty}(\zeta_n^1, \zeta_n^2)$ are correlated (they are equal in the MF case!)
- if $p_n \to 0$ (diluted case), $(\zeta^1, \zeta^2) = \lim_{n\to\infty}(\zeta_n^1, \zeta_n^2)$ are independent.

Idea of proof

The proof follows the usual steps of (i) tightness of the processes $(\eta_n, \hat{\eta}_n)$ and (ii) uniqueness of the limit. The key argument is to control the terms in the expansion of the fluctuation process, the first one being the auxiliary process

$$\hat{\eta}_{n,t} = \frac{1}{n^{3/2}} \sum_{i,j=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) \delta_{(X_t^i, X_t^j)}$$

but really higher-order functionals, indexed by local trees within the graph:

$$C_{n,t}^{<} := \frac{1}{n^3} \sum_{i,j,k=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) \left(\frac{\varepsilon_{i,k}^{(n)(}}{p_n} - 1 \right) \delta_{(X_t^i, X_t^j, X_t^k)},$$

$$C_{n,t}^{\bullet\!-\!\bullet\!-\!\bullet} := \frac{1}{n^3} \sum_{i,j,k=1}^{n} \left(\frac{\varepsilon_{i,j}^{(n)}}{p_n} - 1 \right) \left(\frac{\varepsilon_{j,k}^{(n)}}{p_n} - 1 \right) \delta_{(X_t^i, X_t^j, X_t^k)}$$

Key idea: Grothendieck inequalities

The proof relies on (extensions of) the Grothendieck inequality, whose most simple instance is

Theorem, Grothendieck

There is a universal constant $\mathscr{K}$ such that for any array $a_{i,j}$, for any Hilbert space H,

$$\sup\left\{\left|\sum_{j,k} a_{jk}\langle x_j, y_k\rangle_H\right| : \|x_j\|_H, \|y_k\|_H \leq 1\right\} \leq \mathscr{K} \sup\left\{\left|\sum_{j,k} a_{jk} s_j t_k\right| : s_j = \pm 1,\ t_k = \pm 1\right\}$$

This identity gives an estimate on Sobolev norms of the process $\hat{\eta}_n$ in terms of similar quantities where the Dirac has been replaced by signs in ± 1: we have removed the dependence!

The price we have to pay is that we need to control such weighted sums in the worst-case scenario, which requires stronger concentration estimates: a union bound on the sup gives a factor 4^n so that the previous

$$\mathbb{P}\left(\sup_{s,t}\left|\frac{1}{n^{3/2}}\sum_{i,j=1}^{n}\left(\frac{\varepsilon_{i,j}^{(n)}}{p_n}-1\right)s_i t_j\right| > \frac{1}{n^{\frac{1}{2}-\varepsilon}p_n^{1-\varepsilon}}\right) \leq 2\times 4^n \exp\left(-\frac{n^{\varepsilon}}{4}\right)$$

is no longer summable ! And this does not work for $\hat{\eta}_n$!

In order to deal with $C^{\prec}_{n,t}$, $C^{\overrightarrow{\,}}_{n,t}$, we need higher order inequalities. Bad news: Grothendieck inequalities for multilinear functionals are false in general. Good news: there is a class of functionals for which they remain true [Blei, 2014].
Let $m \geq 1$, $\mathscr{U} = (S_1, \ldots, S_N)$ of non empty sets with $\cup_{i=1}^{N} S_i = \{1, 2, \ldots, m\}$, for $\alpha = (\alpha_j)_{1 \leq j \leq m} \in \mathbb{Z}^m$, define the projections $\pi_{S_i}(\alpha) = (\alpha_j)_{j \in S_i}$. Consider the functional $v_{\mathscr{U}} : l^2\left(\mathbb{Z}^{|S_1|}\right) \times \ldots \times l^2\left(\mathbb{Z}^{|S_N|}\right) \to \mathbb{C}$ defined as $v_{\mathscr{U}}(x_1, \ldots, x_N) = \sum_{\alpha \in \mathbb{Z}^m} x_1(\pi_{S_1}(\alpha)) \cdots x_N(\pi_{S_N}(\alpha))$. Denote, for $1 \leq j \leq m$, by $k_j(\mathscr{U}) = |\{i : j \in S_i\}|$ and by $\mathscr{I}_{\mathscr{U}}$ the minimal incidence $\mathscr{I}_{\mathscr{U}} = \min\{k_j(\mathscr{U}) : j \in \{1, \ldots, m\}\}$.

Theorem

Suppose that $\mathscr{I}_{\mathscr{U}} \geq 2$. Then there exists a positive constant $\mathscr{K}_{\mathscr{U}}$, depending only on the covering $\mathscr{U}$, such that for any finitely supported scalar n-array $a_{j_1 \ldots j_N}$,

$$\sup \left| \left\{ \sum_{j_1, \ldots, j_N} a_{j_1 \ldots j_N} v_{\mathscr{U}}(x_1, \ldots, x_N) : \|x_1\|_{l^2(\mathbb{Z}^{|S_1|})} \leq 1, \ldots, \|x_N\|_{l^2(\mathbb{Z}^{|S_N|})} \leq 1 \right\} \right|$$

$$\leq \mathscr{K}_{\mathscr{U}} \sup \left\{ \left| \sum_{j_1, \ldots, j_N} a_{j_1 \ldots j_N} s_{1,j_1} \cdots s_{N,j_N} \right| : s_{1,j_1} = \pm 1, \ldots, s_{N,j_N} = \pm 1 \right\}. \quad (6)$$

Conclusion

- At the level of the LLN, the mean-field limit remains universal for diffusions on random graphs for both the empirical measure and local empirical measures as long as $np_n \to \infty$.
- These results remains largely true for inhomogeneous connections (W-random graphs)
- At level of the CLT, mean-field fluctuations remain universal as long as the initial condition is independent of the graph, but not be universal in general.
- The optimality of the dilution regime (for now $np_n^4 \to \infty$) remains unclear.

F. Coppini, E. Luçon, and C. Poquet. *Central limit theorems for global and local empirical measures of diffusions on Erdős–Rényi graphs*, June 2022, https://arxiv.org/abs/2206.06655.

Thank you for your attention!