

On a CLT associated

with the intrinsic volumes

of a convex body

Ivan Nourdin,

Univ. of Luxembourg

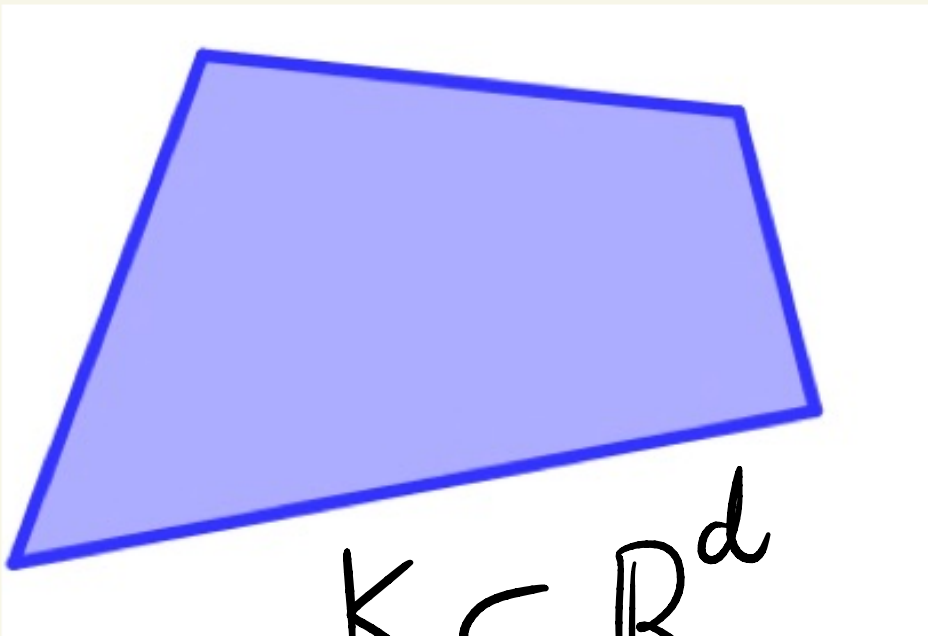
(joint work with Valentin Garino)

Rennes, June 8<sup>th</sup>, 2023

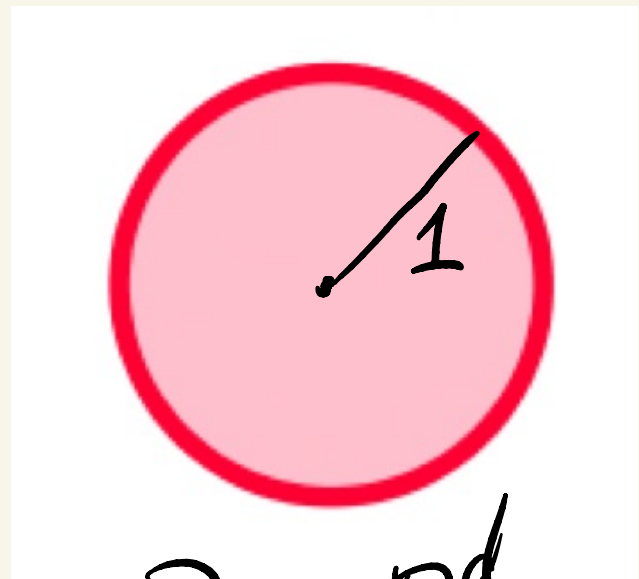
# Heroes of this talk:

Intrinsic volumes of a convex body\*

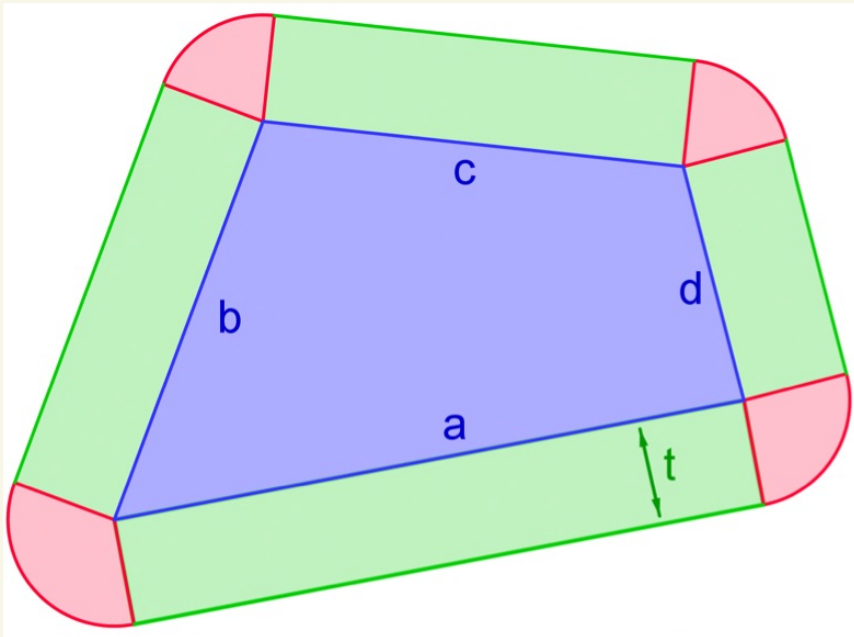
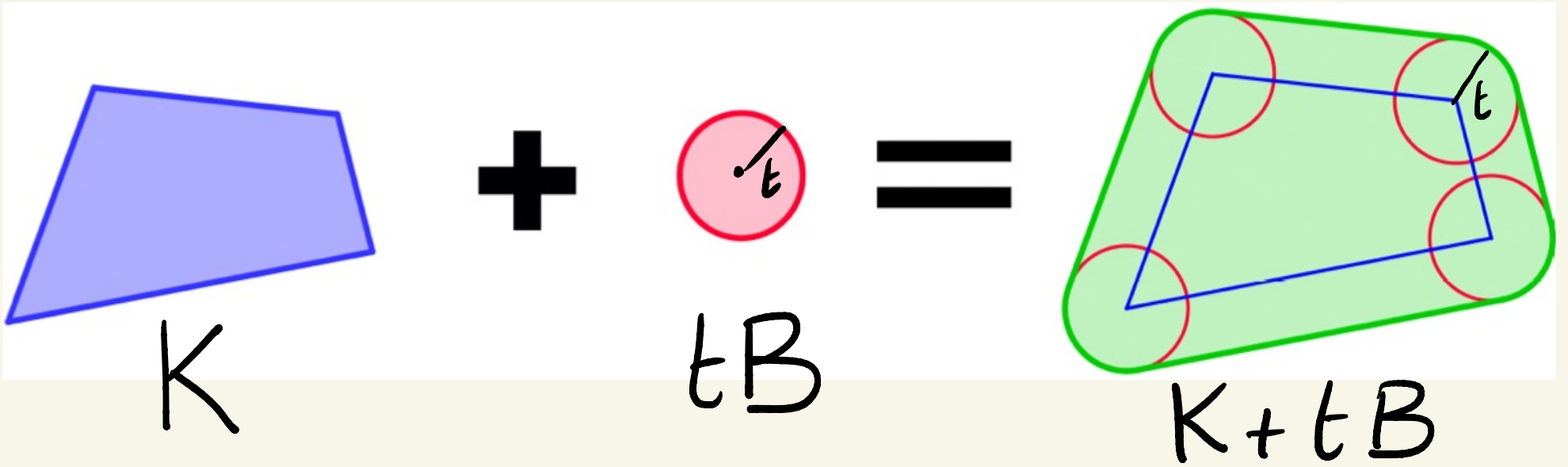
(\* compact convex set with non-empty interior)



$K \subset \mathbb{R}^d$   
convex body



$B \subset \mathbb{R}^d$   
unit ball



$$\begin{aligned}
 & \text{Vol}(K + tB) \\
 &= \text{Vol}(K) \\
 &+ t \text{Per}(K) \\
 &+ \pi t^2
 \end{aligned}$$

# Steiner formula in $\mathbb{R}^d$

$\text{Vol}(K + tB)$  is a polynomial in  $t$  of degree  $d$

$$\text{Vol}(K + tB) = \sum_{k=0}^d a_k(K) t^k$$

$(v_k(K))_{0 \leq k \leq d}$

intrinsic volumes of  $K$

$$k! \times v_{d-k}(K)$$

$$= \frac{\pi^{k/2}}{\Gamma(1+k/2)} = \text{volume of unit ball } \subset \mathbb{R}^k$$

Intrinsic volumes measure how "big" is  $K$ .

# Commonly encountered intrinsic volumes

$$V_0(K) = 1$$

$$V_1(K) = \text{intrinsic width} \quad (\text{ex: } V_1(B) \sim \sqrt{2\pi d})$$

⋮

$$V_{d-1}(K) = \frac{1}{2} \times \text{surface area of } K$$

$$V_d(K) = d\text{-dim. volume of } K$$

will be used later on!

Some properties satisfied by intrinsic volumes:

- $V_k(K) \geq 0$  [positivity]
- $V_k(K \subset L) \Rightarrow V_k(K) \leq V_k(L)$  [↑]
- $V_k(tK) = t^k V_k(K)$  [homog.]
- $V_k(K \times \{0\}) = V_k(K)$  [intrinsic]  
↑  
"raison d'être" of  $V_k$ !
- $V_k(K \cup L) + V_k(K \cap L) = V_k(K) + V_k(L)$  [valuation]

Many nice (and deep) inequalities are available for intrinsic volumes

Ex: The isoperimetric inequality states that "among all sets  $A \subset \mathbb{R}^d$  with a given volume, the ball has the minimal surface area." Otherwise stated:

if  $r \geq 0$  is s.t.  $V_d(K) = V_d(rB)$  then  $V_{d-1}(K) \geq V_{d-1}(rB)$ .

$$r = \left( \frac{V_d(K)}{V_d(B)} \right)^{\frac{1}{d}}$$

$$r \leq \left( \frac{V_{d-1}(K)}{V_{d-1}(B)} \right)^{\frac{1}{d-1}}$$

More generally [Alexandrov]

$$\left( \frac{V_j(K)}{V_j(B)} \right)^{\frac{1}{j}} \leq \left( \frac{V_i(K)}{V_i(B)} \right)^{\frac{1}{i}}$$

$$V_j \geq i$$

Chevet, Mc Mullen (independently):

$\{j! v_j(K)\}_{0 \leq j \leq d}$  is log-concave

↓  
[( $x_j$ ) LC if  $x_j^2 \geq x_{j+1} x_{j-1} \forall j$ ]

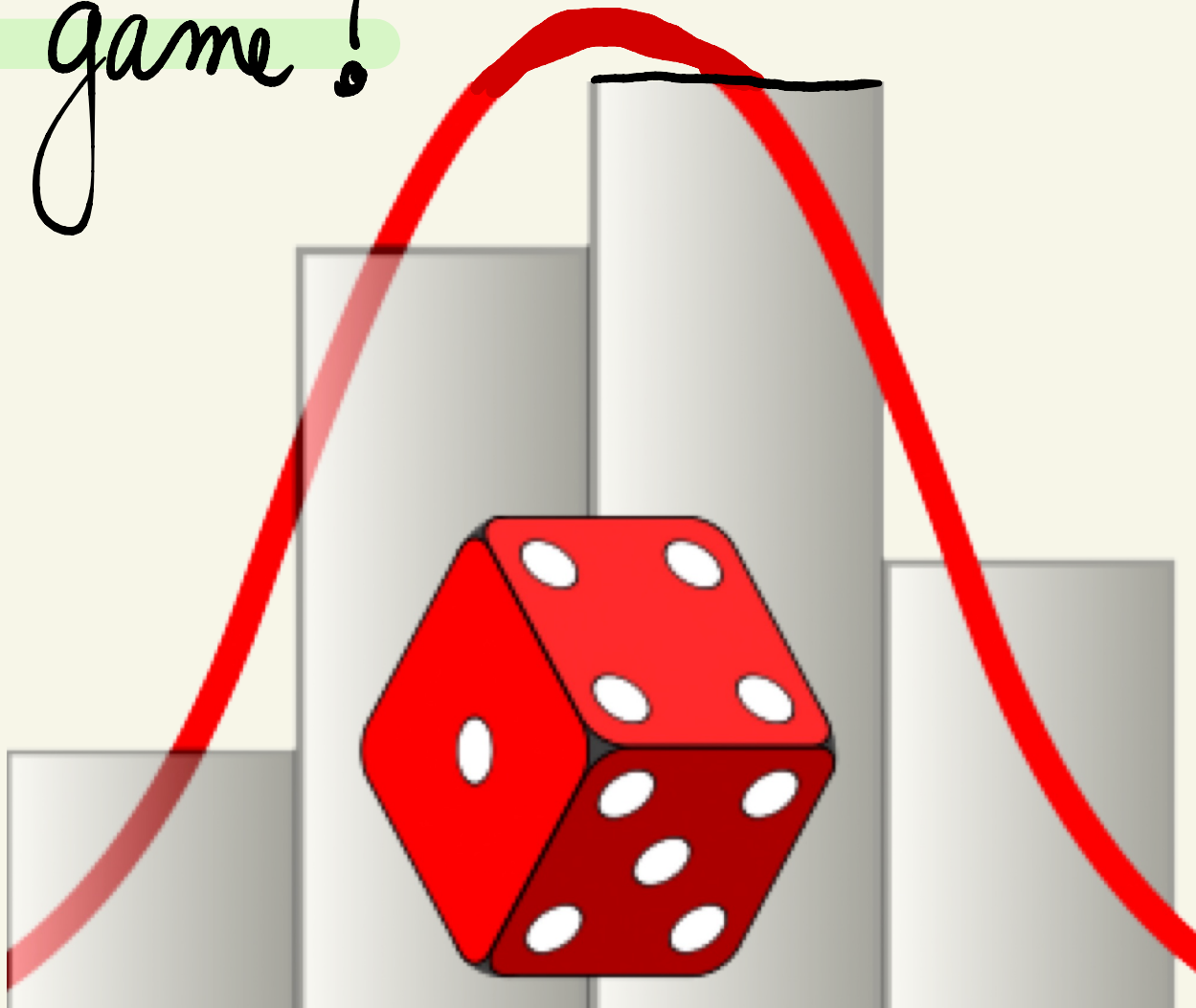
$\Leftrightarrow \{v_j(K)\}_{0 \leq j \leq d}$  is ultra log-concave

↓  
[( $x_j$ ) ULC if  $j x_j^2 \geq (j+1) x_{j+1} x_{j-1} \forall j$ ]

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I won't need any other geometric properties.

Now, let's probability  
enter the game!





Set  $W(K) = \sum_{k=0}^d v_k(K) \geq 0$

for Wills  $(= \int_{\mathbb{R}^d} e^{-\pi \text{dist}^2(x, K)} dx)$

Consider  $Z_K: \Omega \rightarrow \mathbb{R}_+$  distributed as

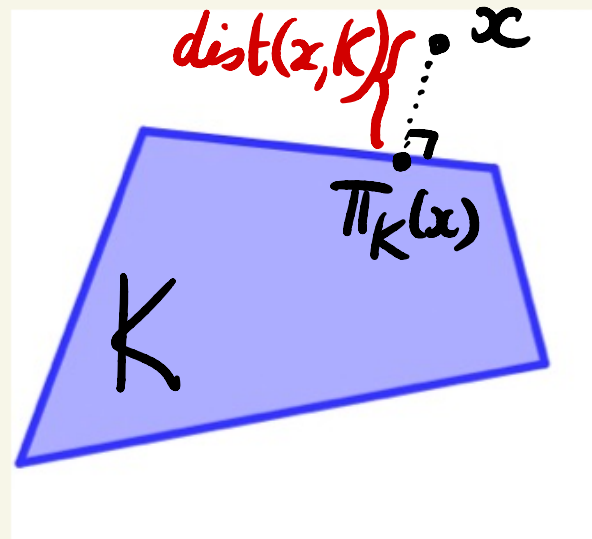
$$P(Z_K = k) = \frac{v_k(K)}{W(K)}, \quad 0 \leq k \leq d$$

Qu: Asymptotic behavior of  $Z_K$  as  $d \rightarrow \infty$ ?

Ex:  $K = [0, 1]^d \rightsquigarrow v_k(K) = \binom{d}{k} \rightsquigarrow Z_K$  satisfies a CLT.

FACT:

$$\begin{aligned} \text{dist}^2(x, K) \\ = \|x - \pi_K(x)\|^2 \end{aligned}$$



In Lotz, McCoy, Nourdin, Peccati, Tropp (2019) : Steiner  $\Leftrightarrow$

$$\begin{aligned} & \int_{\mathbb{R}^d} g(\pi \text{dist}^2(x, K)) e^{-\pi \text{dist}^2(x, K)} dx \\ &= g(0) v_d(K) + \sum_{j=0}^{d-1} \frac{v_j(K)}{\Gamma(\frac{d-j}{2})} \int_0^\infty g(r) r^{-1+\frac{d-j}{2}} e^{-r} dr \end{aligned} \quad (\text{LM+})$$

For  $g \equiv 1$ , we recover Wills' result:  $\int_{\mathbb{R}^d} e^{-\pi \text{dist}^2(x, K)} dx = W(K)$

Let  $X_K: \Omega \rightarrow \mathbb{R}^d$  have the (log-concave) density

$$\frac{1}{W(K)} e^{-\pi \text{dist}^2(x, K)}, \quad x \in \mathbb{R}^d.$$

Define  $H_K: \Omega \rightarrow \mathbb{R}_+$  as

$$H_K := \pi \text{dist}^2(X_K, K)$$

Hadwiger-Wills  
information content of  $X_K$

Law of  $H_K$  is equivalent to Law of  $Z_K$

(continuous)  
(log-concave)

(discrete)

QU: Asymptotic behavior of  $H_K$ ?

Lemma (probabilistic version of  $LM+$ )

$$H_k \stackrel{(L)}{=} \sum_{j=1}^{d-Z_k} \gamma_j$$


where  $\gamma_j \sim \Gamma(\frac{1}{2}, 1)$  are  $\perp$  (Rk.  $\sum_{j=1}^k \gamma_j \sim \Gamma(\frac{k}{2}, 1)$ )

$\perp$   $\dots P(Z_k = j) = \frac{v_j(k)}{W(k)}, \quad 0 \leq j \leq d$

Prop

If  $\frac{\text{Var}(Z_k)}{d - E(Z_k)} \approx 0$

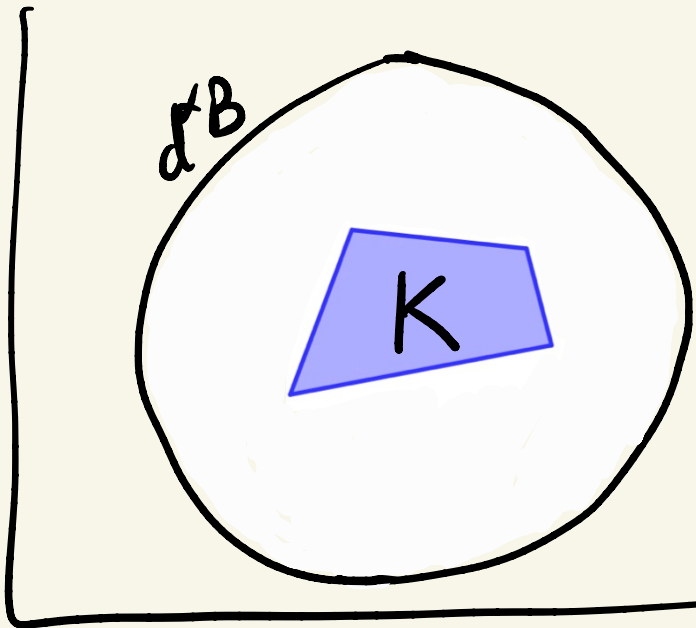
then

$$\frac{H_k - EH_k}{\sqrt{\text{Var} H_k}} \approx N(0, 1)$$


LM (Garino, N.) If  $K \subset d^B$  with  $0 < \alpha < \frac{1}{2}$  then

$$\frac{\text{Var}(Z_K)}{d - \mathbb{E}(Z_K)} \approx 0$$

(as  $d \rightarrow \infty$ )



Proof: • Set  $\alpha_j = \frac{v_j(K)}{w(K)}$ .  $(\alpha_j)$  is **ULC**, i.e.  $j \alpha_j^2 \geq (j+1) \alpha_{j-1} \alpha_{j+1}$

$$\Leftrightarrow \frac{1}{j+1} \frac{\alpha_j}{\alpha_{j+1}} \geq \frac{1}{j} \frac{\alpha_{j-1}}{\alpha_j}$$

We deduce

$$\frac{\alpha_j}{\alpha_{j+1}} - \frac{\alpha_{j-1}}{\alpha_j} \geq \left(\frac{j+1}{j} - 1\right) \frac{\alpha_{j-1}}{\alpha_j} = \frac{1}{j} \frac{\alpha_{j-1}}{\alpha_j} \geq \frac{\alpha_0}{\alpha_1}$$

Summing (dominos ☺):  $\frac{\alpha_j}{\alpha_{j+1}} - \frac{\alpha_0}{\alpha_1} \geq j \frac{\alpha_0}{\alpha_1} \Rightarrow \frac{\alpha_j}{\alpha_{j+1}} \geq (j+1) \frac{\alpha_0}{\alpha_1} = \frac{j+1}{v_1(K)}$

$$\Rightarrow \mathbb{E}(Z_K) = \sum_{j=0}^{d-1} (j+1) \alpha_{j+1} \leq v_1(K) \sum_{j=0}^{d-1} \alpha_j \leq v_1(K)$$

•• For the variance (Johnson) let  $(x_j) \subset \mathbb{R}_+$  be such that  $\sum x_j = 1$ .

Let  $Z: \Omega \rightarrow \mathbb{R}_+$  be distributed according to  $P(Z=j) = x_j$ .

If  $(x_j)$  is  $c$ -logconcave (that is,  $\frac{x_j}{x_{j+1}} - \frac{x_{j-1}}{x_j} \geq c$ )

[discrete Bakery-Emery]

then  $\text{Var}(f(Z)) \leq \frac{1}{c} \sum_{j=0}^{\infty} x_j (f(j+1) - f(j))^2$ .

In particular:  $\text{Var}(Z) \leq \frac{1}{c}$

Cor  $\text{Var}(Z_K) \leq v_1(K)$

Proof:  $\left(\frac{v_j(K)}{w(K)}\right)$  is  $\frac{1}{v_j(K)}$ -logconcave & Johnson  $\Rightarrow$  CQFD.

## Summary:

We assume  $K \subset d^\alpha B$  with  $\alpha < \frac{1}{2}$

We need to prove  $\frac{\text{Var}(Z_K)}{d - \mathbb{E}Z_K} \approx 0$

We have shown that  $\begin{cases} \mathbb{E}Z_K \leq v_1(K) \\ \text{Var} Z_K \leq v_1(K) \end{cases}$

...  $K \subset d^\alpha B$  with  $\alpha < \frac{1}{2}$

$$\Rightarrow v_1(K) \leq v_1(d^\alpha B) \stackrel{\text{homogeneity}}{=} d^\alpha v_1(B) \underset{d \rightarrow \infty}{\sim} \sqrt{2\pi} d^{\alpha + \frac{1}{2}}$$

$$\Rightarrow \mathbb{E}(Z_K) = o(d) \Rightarrow d - \mathbb{E}(Z_K) \sim d$$

$$\Rightarrow \frac{\text{Var}(Z_K)}{d - \mathbb{E}(Z_K)} = O\left(d^{\alpha + \frac{1}{2} - 1}\right) = O\left(d^{\alpha - \frac{1}{2}}\right) \longrightarrow 0$$

COR. If  $K \subset d^\alpha B$  with  $\alpha < \frac{1}{2}$

then  $\frac{H_K - \mathbb{E}H_K}{\sqrt{\text{Var}H_K}} \xrightarrow{\mathcal{L}} N(0,1)$

Qu: Can we prove convergence in TV?

→ Stein's method!



Notation:  $X \sim e^{-\phi(x)} dx$  with  $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$

$$Y = \nabla \phi(X)$$

( $\hat{Y}$  ind. copy /  $Y_t = e^{-t} Y + \sqrt{1-e^{-2t}} \hat{Y}$ )

$$H: \mathbb{R}^d \rightarrow \mathbb{R} \rightsquigarrow F = \frac{H(Y) - \mathbb{E}H(Y)}{\sqrt{\text{Var} H(Y)}} = \sigma^2$$

Prop:  $d_{TV}(F, N(0,1)) \leq A + B$  where

$$A = \frac{CST}{\sigma^2} \sqrt{\text{Var} \left( \int_0^\infty e^{-t} \langle (\text{Hess} \phi)(X) \nabla H(Y), \mathbb{E}_\infty(\nabla H(Y_t)) \rangle dt \right)}$$

$$B = \frac{CST}{\sigma^2} \int_0^\infty e^{-2t} \sqrt{\mathbb{E} \left[ \text{Tr} \left( ((\text{Hess} \phi)(X) - (\text{Hess} \phi)(X_\infty)) (\text{Hess} H)(Y_t) \right)^2 \right]} dt$$

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1<sup>st</sup> case:  $\phi(x) = \frac{1}{2} \|x\|^2 + \frac{d}{2} \ln(2\pi)$ , that is  $X \sim N(0, I_d)$

$Y = X$  and  $F = \frac{H(X) - \mathbb{E}H(X)}{\sqrt{\text{Var} H(X)}}$ . In this case:

$$A = \frac{CST}{\sigma^2} \sqrt{\text{Var} \left( \int_0^\infty e^{-t} \langle \nabla H(Y), \mathbb{E}_\infty(\nabla H(Y_t)) \rangle dt \right)}$$

$$B = 0$$

Malliavin  
-Stein

$\text{Var} \langle DF, -DL^*F \rangle$

Prop:  $d_{TV}(F, N(0,1)) \leq A + B$  where

$$A = \frac{CST}{\sigma^2} \sqrt{\text{Var}\left(\int_0^\infty e^{-t} \langle (\text{Hess}\phi)(X) \nabla H(Y), E_\infty(\nabla H(Y_t)) \rangle dt\right)}$$

$$B = \frac{CST}{\sigma} \int_0^\infty e^{-2t} \sqrt{E\left[\text{Tr}\left((\text{Hess}\phi)(X) - (\text{Hess}\phi)(X_\infty)\right) (\text{Hess} H)(Y_t)\right]^2} dt$$

2<sup>nd</sup> case [Our case!]  $H(x) = \|x\|^2$

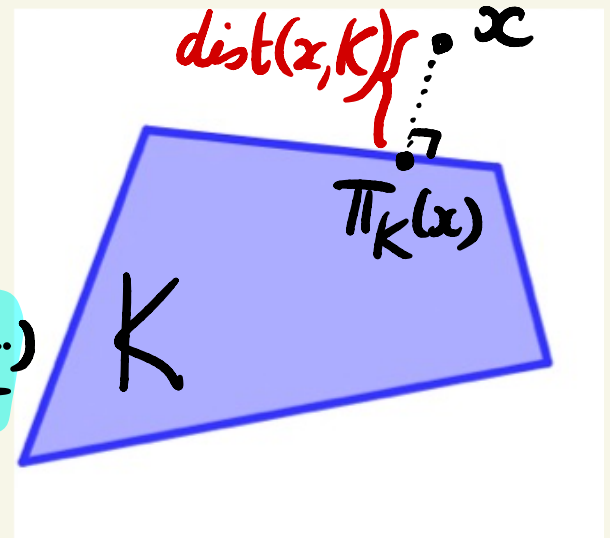
$$\hookrightarrow \phi(x) = \pi \text{dist}^2(x, K) + \ln(w(K))$$

$$y = \nabla \phi(x) = \pi(x - \pi_K(x))$$

$$H(\nabla \phi(x)) = \pi^2 \text{dist}^2(x, K) \rightsquigarrow F = \frac{\text{dist}^2(X_K, K) - E(\cdot)}{\sigma(\cdot)}$$

$$A = \frac{CST}{\sigma^2} \sqrt{\text{Var}\left(\int_0^\infty e^{-t} \langle Y, E_\infty(Y_t) \rangle dt\right)}$$

$$B \leq \frac{CST}{\sigma} \sqrt{\text{Var}(\text{Tr}(\text{Hess}\phi(x)))}$$



THM. (Garino, Nourdin)

If  $K \subset d^\alpha B$  with  $\alpha < \frac{1}{2}$

then  $d_{TV}\left(\frac{H_K - \mathbb{E}H_K}{\sqrt{\text{Var } H_K}}, N(0, 1)\right) = O(d^{\alpha - \frac{1}{2}})$

Proof: Second order Brascamp-Lieb inequality  
& (many) other things!

Open question: do we have a CLT for  $Z_K$ ?

(In LM+, we studied concentration inequalities)

END!