Optimal switching problems with an infinite set of modes: an approach by randomization

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A Backward stochastic excursion with Ying Hu
in honor of his 60th birthday
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Outline of the talk

I- Preliminaries & motivations

- The Optimal Switching problem (OSP): primal formulation.
- Motivation for the "randomization" (dual) approach.
- The dual formulation.

II- The two main results

(i) equality between the two value functions;
(ii) new BSDE characterization.
(iii) Comments & ideas of proof.

III- Conclusion & perspectives

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I.1 Primal optimal switching problem and value function

On a standard prob. space $(\Omega, \mathbb{F}, \mathbb{P})$, let

- W: standard d-dim. Brownian Motion, W $\mathbb{F}$-adapted. usually: $\mathbb{F} = \mathcal{F}^W \vee \mathcal{N}$.
- T fixed finite horizon; A set of modes (possibly **infinite**).
- $\forall\,(x_0, e) \in \mathbb{R}^n \times A$, let X^e (path dependent) proc. s.t.

$$\forall\, t \in [0, T], \quad X_t^e = x_0 + \int_0^t \left(b^e(s, X_\cdot^e)ds + \sigma^e(s, X_\cdot^e)dW_s\right),$$

Motivations

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$$\forall\, t \in [0, T], \quad X^e_t = x_0 + \int_0^t \left(b^e(s, X^e_\cdot)ds + \sigma^e(s, X^e_\cdot)dW_s \right),$$

Let $(f^e)_e$, $(g^e)_e$ and $(c_{e,e'})_{(e,e')}$: 3 families of (possib. **random**) real-valued data

(i) $f^e(s, X_\cdot)$: instant. profit (when system in mode e)

(ii) $g^e(X_\cdot)$: payoff at time T when syst. in mode e,

(iii) $c_{e,e'}(s, X_\cdot)$: *nonnegative* penalty costs at time s.

I.1 Primal optimal switching problem and value function

1. Let $\alpha = (\tau^n, \xi^n)_{n\geq 1}$ a *management* strategy s.t.
   (i) τ^n increas. seq. of $\mathbb{R}^+$ valued $\mathbb{F}$ -stopping times;
   (ii) for each n, ξ^n both A-valued and $\mathcal{F}^W_{\tau_n}$-meas.

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   (ii) for each n, ξ^n both A-valued and $\mathcal{F}^W_{\tau_n}$-meas.
2. To α, we associate the state proc. a as follows

$$a_s = \xi^1 \mathbf{1}_{s<\tau_1} + \sum_{n\geq 1} \xi^{n+1} \mathbf{1}_{\tau^n \leq s < \tau_{n+1}} \mathbf{1}_{\tau^n < T}$$

a: piecewise constant proc. A-valued

By abuse, one may replace α by a.

I.1 Primal value function: Admissible set $\mathcal{A}$

$a = (\tau^n, \xi^n)$ is said *admissible* i.e. a in $\mathcal{A}_t$ if

H$_1$ $(\tau^n(\cdot), \xi^n(\cdot))_n$-$\mathbb{R}^+ \times A$-valued $\mathbb{F}$-adapt. s. t.
$\tau_n(\omega) \to +\infty$ and $\tau^n < \tau^{n+1}$, $\mathbb{P}$-a.s and $\tau_0 = t$.
a is in $\mathcal{A}_t^e \subset \mathcal{A}_t$ if $\xi_0 = e$ (at $t = \tau_0$).
simultaneous switchings prohibited, i.e.

$$\forall (a_1, a_2, a_2) \in A^3, \quad c_{a_1,a_2}(t,x) + c_{a_2,a_3}(t,x) > c_{a_1,a_3}(t,x)$$

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H₂ **H₁** implies: $N_T^a(\omega) = \text{Card}\{\tau^n(\omega), \tau^n < T\} < +\infty$, $\mathbb{P}$-a.s

H₃ Impose $\tau^n \neq T$: **no switching at terminal time.**
In finite case, equivalent to:

$$\forall\ (i,j) \in A \times A, \ \ g^i(x) > g^j(x) - c_{i,j}(T,x).$$

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I.1. Primal optimal switching problem (OSP) and value function

1. For a in $\mathcal{A}$, X^α (or X^a) the controlled proc. s.t.

$$dX^a = b^a(s, X^a)ds + \sigma^a(s, X^a)dW_s$$

with $b^a(s,x) = b^{\xi_0}(s,x)\mathbf{1}_{s<\tau^1} + \sum_{n\geq 1} b^{\xi^n}(s,x)\mathbf{1}_{\tau^n\leq s<\tau^{n+1}}$.

Similar definition for $\sigma^a(s,x)$.

Remark: b and σ path-dependent $\Rightarrow$ X^a no more Markovian (PDE approach not available).

I.1. Primal OSP: Technical assumptions (1)

- Mathematical assumptions:
  - A: Borel set (example: any subspace of $\mathbb{R}^d$);
  - Both $(b^e, \sigma^e)_e$, (f^e, g^e), $(c_{e,e'})_{e,e'}$ may be path-dependent;
  - Let $\mathbb{C}^n$: set of continuous paths $(s \mapsto x(s))_{s\in[0,T]}$ Topology on $\mathbb{C}^n$: $|x|_* = \sup_{s\in[0,T]} |x(s)|$
  - Measurability $(t, \omega, e) \mapsto b^e(t, x(\omega), \omega)$, $\sigma^e(t, \omega, x(\omega), e)$ are $Prog(\mathbb{C}^n) \otimes \mathcal{B}(A)$ meas.; (similar for $f^e, g^e, c_{e,e'}$) $Prog(\mathbb{C}^n)$: σ-algebra of prog. measurable maps on $[0, T] \times \Omega$.

I.1. Primal OSP: Technical assumptions (2)

(i)) Continuity of data w.r.t (x, e)

- For every t in $[0, T]$,
$(x, e) \mapsto b_t(x, e)\ \sigma_t(x, e),\ \ f_t(x, e), g(x, e)$ are continuous on $\mathbb{C}^n \times A$ $(x, e, e') \mapsto c_t(x, e, e')$ is continuous on $\mathbb{C}^n \times A \times A$.

- Regularity & growth assumpt (wrt x):
$\exists\ K > 0$ s.t. $\forall\ (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A$,

(ii) $|b_t(x, e) - b_t(x', e)| + |\sigma_t(x, e) - \sigma_t(x', e)| \leq K|x - x'|_{t*}$
Similar for other data.

(iii) $|b(t, 0, e)| + |\sigma(t, 0, e)| \leq K$;

I.1. Primal OSP: Technical assumptions (3)

• Growth assumpt wrt x (cont')
$\exists\ r,\ K > 0$ s.t. $\forall\ (t, x, x', e, e') \in [0, T] \times \mathbb{C}^n \times \mathbb{C}^n \times A \times A$,
(iii) $|f(t, x, e| + |g(x, e)| + |c(t, x, e, e')| \leq K\big(1 + |x|^r_{t*}\big)$.

Comment: Thanks to those assumptions, one gets
(a) Strong estim. of the moments of proc. X^e
(Cosso-Confortola-Fuhrman '18);
(b) Estimate for the value functional

I.1 Primal control problem: the value function

1. Fixing e in A, let $\mathcal{V}^e$ be the primal value function s.t.

$$\mathcal{V}^e = \sup_{\alpha \in \mathcal{A}^e} \big(J(\alpha)\big), \text{ where}$$

$$J(\alpha) \quad = \mathbb{E}\left(g^{a_T}(X_.) + \int_0^T f^{a_s}(s, X^a_.)ds - \sum_{\substack{n \geq 1, \\ \tau_n < T}} c_{\xi_{n-1},\xi_n}(\tau^n, X^a_{\tau^n}) \right)$$

I.1 OSP with finite set of modes- BSDE characterization

Under *appropriate* assumptions on data $(f^e, g^e, (c_{e,e'})$ with $e,\ e' \in \mathcal{J} = \{1, \cdots, m\}$
there exists a solution $(Y^e, Z^e, K^e)_{e\in\mathcal{J}}$ to the BSDE system

$$\begin{cases} Y_t^e = & g^e(X_T) + \int_t^T f_s^e(X_s)\,ds + K_T^e - K_t^e \\ & \quad - \int_t^T Z_s^e dW_s, \\ Y_s^e & \geq \max_{\{e'\in\mathcal{J}\setminus\{\ e\}\}} \Big(Y_s^{e'} - c_{e,e'}(s, X_s) \Big) \text{ and} \\ \int_0^T (Y_s^e & - \max_{\{e'\in\mathcal{J}\setminus\{\ e\}\}} \Big(Y_s^{e'} - c_{e,e'}(s, X_s) \Big)) dK_s^e = 0 \end{cases}$$

s.t. $Y_0^e = \mathcal{V}^e$.

A (non exhaustive) review of the literature

(1) **OSP with finite set of modes:**

- (i) Using PDE approaches: Ishii-Koike '91, Yong-Zhou '99, Ludkowski '05, Carmona-Ludkovski '07-08, ...
- (ii) Using BSDE and analyt. tools: Hamadène-Jeanblanc '02, Djehiche-Hamadene-Popier '08, Hamadène Zhang '10, Hu-Tang '10, Chassagneux-Elie -Kharroubi '11; Elie-Kharroubi '10, '14 ...
- (iii) Standard OSP with refinements: infinite horizon, Levy driven case (forward diff), partial information, non positive costs (Lundstrom-Olofsson, R. Martyr, B. El Asri) ..

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- (iii) Standard OSP with refinements: infinite horizon, Levy driven case (forward diff), partial information, non positive costs (Lundstrom-Olofsson, R. Martyr, B. El Asri) ..

(2) **Connection between "finite" OSP & constrained BSDE:**

(a) Ma-Pham-Kharroubi '10 (Markovian setting)

(b) Elie-Kharroubi ('14) (Non Markovian case)

I.3. Why choosing the "randomization" method ?

1. when A infinite, the (*a priori infinite*) system of RBSDEs does not seem well posed (at least to us...)
2. BSDE charact. of primal OSP: many ingredients use the finiteness of A.

Preliminaries to our result: comments

I.3. Why choosing the "randomization" method ?

1. when A infinite, the (*a priori infinite*) system of RBSDEs does not seem well posed (at least to us...)
2. BSDE charact. of primal OSP: many ingredients use the finiteness of A.
3. Randomization allows to tackle general cases: path-dependency, degenerate diffusions, case of an infinite set of modes.
4. Another motivation: in the case of finite OSP, connection already proved by Elie & Kharroubi.

I.2 Randomized set-up & dual formulation

1. On $(\Omega', \mathbb{F}', \mathbb{P}')$ $\mu = (\sigma^m, \zeta^m)_m$: Poisson random meas. s.t.
   (i) the marks $(\sigma^m, \zeta^m)_m$ are $\mathbb{R}^+ \times A$-valued;
   (ii) μ **indep.** of W with $\bar{\mu}(de, ds) = \lambda(de)ds$ and
   λ intensity meas. with **full support** on A and $\lambda(\mathbf{A}) < +\infty$.

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   λ intensity meas. with **full support** on A and $\lambda(\mathbf{A}) < +\infty$.

2. The *randomized* dual set up := $(\hat{\Omega}, \hat{\mathbb{P}}, \hat{\mathcal{F}}, \hat{W}, \hat{\mu})$:
   (2.i) Let $\hat{\Omega} := \Omega \times \Omega'$, $\hat{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$ and $\hat{\mathcal{F}} = \mathbb{F}^{W,\mu}$, with

$$\mathbb{F}^{W,\mu} := (\mathbb{F}^W \vee \mathbb{F}^\mu) \vee \mathcal{N}$$

   (2.ii) $\hat{W}(\omega, \omega') = W(\omega)$ remains a $\mathbb{F}^{W,\mu}$- Brownian motion;
   $\hat{\mu} := (\hat{\sigma}^m, \hat{\zeta}^m)_m$ Poisson r.m. with $\mathbb{F}^{W,\mu}$-prog. meas random marks and **same determ. compensator** $\hat{\mu}$.

I.2. The randomized set-up and dual formulation

1. Let I (resp. $\hat{I}$) the Poisson point proc. assoc. with μ (resp. $\hat{\mu}$) as follows

$$\forall\ t \in [0, T], \quad I_t = \zeta^0 \mathbf{1}_{t<\sigma^1} + \sum_{m\geq 1} \zeta^m \mathbf{1}_{\sigma^m \leq t < \sigma^{m+1}}.$$

Note that $N_T^I := \mathrm{Card}\{m \geq 1,\ \sigma_m(\omega') < T\} < \infty$, $\mathbb{P}'$-a.s.

2. On *randomized* prob. space, $(\hat{I}, X^{\hat{I}})$ is a **forward uncontrolled proc.** with

$$X_t^{\hat{I}} = x_0 + \int_0^t \left(b^{\hat{I}_s}(s, X^{\hat{I}})ds + \sigma^{\hat{I}_s}(s, X^{\hat{I}})dW_s\right)$$

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I.2. The randomized set-up and dual formulation

1. To any proc. $\hat{\nu}$ $\mathbb{F}^{W,\mu}$-meas., let $\kappa^{\hat{\nu}}$ the Doleans-Dade proc.

$$\kappa_T^{\hat{\nu}} = \mathcal{E}_T((\hat{\nu} - 1) \star \tilde{\mu}) = e^{-\int_0^T \int_A (\hat{\nu}_s(e)-1)\lambda(de)ds} \prod_{\substack{m\geq 1 \\ \zeta_m < T}} \hat{\nu}_{\sigma^m}(\zeta^m)$$

2. Let $\hat{\mathbb{P}}^{\hat{\nu}}$ with density $\kappa^{\hat{\nu}}$, i.e. $\dfrac{d\hat{\mathbb{P}}^{\hat{\nu}}}{d\hat{\mathbb{P}}} = \kappa^{\hat{\nu}}$

   then, under $\hat{\mathbb{P}}^{\hat{\nu}}$,
   (a) $\hat{I}$ remains Poisson point proc.;
   (b) its new compensated meas. $\hat{\nu}_s(e)\lambda(de)ds$
3. Set of dual controls

$$\mathcal{A}^R := \{\hat{\nu} : \hat{\Omega}\times[0,T]\times A \mapsto]0;\infty[\text{ meas. and essentially bounded}\}$$

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I.2 The randomized set-up: dual formulation

1. Let $\mathcal{V}_0^R = \sup_{\hat{\nu} \in \mathcal{A}^R} J^{\mathcal{R}}(\hat{\nu})$ be the dual value function with

$$J^R(\hat{\nu}) = \underbrace{\hat{\mathbb{E}}^{\hat{\nu}}\left(g(X^I, I_T) + \int_t^T f(s, X^I, I_s) ds \right)}_{=J_1^R(\hat{\nu})} - \underbrace{\hat{\mathbb{E}}^{\hat{\nu}}\left(\sum_{m \geq 1} c_{\zeta_{m-1}, \zeta_m}(\sigma^m, X_{\sigma^m}) \right)}_{=J_2^R(\hat{\nu})}$$

$\hat{\mathbb{E}}^{\hat{\nu}}$ stands for expectation under meas. $\hat{\mathbb{P}}^{\hat{\nu}}$.

II. Theorem 1

Under all previous assumptions on the primal & dual version of the OSP, one obtains

$$\mathcal{V}_0 = \mathcal{V}_0^{\mathcal{R}} = v_0(x, e).$$

This common value function only depends on $X_0 = x$ and initial mode e and not of the choice of the randomized set up: (i.e. **neither on the construction of the extended dual set-up nor on the choice of intensity measure λ).**

II. Theorem 2: BSDE characterization

The following BSDE (with constrained jumps)

$$\begin{cases} Y_t^{\mathcal{R}} = g(X, I_T) + \int_t^T f_s(X, I_s)\, ds + K_T - K_t \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_{(t,T]} \int_A U_s(e)\, \mu(ds\, de), \\ U_t(e) \le c_t(X, I_{t-}, e), \quad \textit{(non-linear jump constraint)} \end{cases} \tag{1}$$

admits a *minimal* solution $Y^{\mathcal{R}}$ such that

$$Y_0^{\mathcal{R}} = \mathcal{V}_0^{\mathcal{R}}.$$

Remark: (**??**) is a BSDE with constrained jumps.
(i) K non decreas. predic. proc s.t. K **only càdlàg** in general.
(ii) $Y_t^{\mathcal{R}}$ is $\mathcal{F}_t^{W,\mu}$-adapted.

Main results: comments

Connection with BSDE in the case of finite set of modes (Elie-Kharroubi '14)

Let $\mathcal{J}$ set of modes and let $(Y^e)_{e\in\mathcal{J}}$ solving

$$\begin{cases} Y_t^e = & g(e, X_T) + \int_t^T f_s^e(X_s)\,ds + K_T^e - K_t^e \\ & - \int_t^T Z_s^e dW_s, \\ Y_s^e & \geq \max_{\{j\in\mathcal{J}\setminus\{\,i\}\}}\left(Y_s^j - c_{e,j}(s, X_s)\right) \text{ and} \\ \int_0^T (Y_s^e - \max_{\{j\in\mathcal{J}\setminus\{\,e\}\}}\left(Y_s^j - c_{e,j}(s, X_s)\right) dK_s^e = 0 \end{cases} \tag{2}$$

If both the dual BSDE (**??**) and BSDE system (**??**) have a solution then

$$Y_t^{\mathcal{R}} = Y_t^{I_t} \text{ and } U_t(e) = Y_t^e - Y_t^{I_{t-}}.$$

Main results: new BSDE characterization

The new BSDE representation

Let $Y^{\mathcal{R}}$ be the *minimal* solution of following BSDE

$$\begin{cases} Y_t = g(X, I_T) + \int_t^T f_s(X, I_s)\,ds + K_T - K_t \\ \qquad\qquad - \int_t^T Z_s dW_s - \int_{(t,T]}\int_A U_s(e)\,\mu(ds\,de), \\ U_t(e) \le c_t(X, I_{t-}, e),\ \lambda(de)dsd\mathbb{P}-\text{a.e.} \end{cases} \tag{3}$$

Since one has

$$Y_0^{\mathcal{R}} = \mathcal{V}_0^{\mathcal{R}} = \sup_{\nu \in \mathcal{A}^R} J^{\mathcal{R}}(\nu),$$

then, combining with first main result

$$Y_0^{\mathcal{R}} = \mathcal{V}_0^{\mathcal{R}} = \mathcal{V}_0 = \sup_{\alpha \in \mathcal{A}} \mathcal{J}(\alpha).$$

Theorem 2: sketch of proof (1/3)

1. Existence by penalization: let $(Y^n,\ Z^n,\ U^n)$ solve

$$Y_t^n = g(X, I_T) + \int_t^T f_s(X, I_s)\, ds + K_T^n - K_t^n$$
$$- \int_t^T Z_s^n\, dW_s - \int_t^T \int_A U_s^n(a)\, \mu(ds\, da),$$

where $K_t^n = n \int_0^t \int_A \left(U_s^n(e) - c_s(X, I_{s-}, e)\right)^+ \lambda(de)\, ds.$

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where $K_t^n = n\int_0^t \int_A \big(U_s^n(e) - c_s(X, I_{s-}, e)\big)^+ \lambda(de)\,ds$.

2.a Probab representation of Y^n

$$Y_t^n = \text{ess sup}_{\nu\in\mathcal{V}_n} \ \mathbb{E}^\nu\Big(g(X, I_T) + \int_t^T f_s(X, I_s)\,ds - \sum_{l\geq 1} 1_{t<\sigma_l<T}\, c_{\sigma_l}(X, \eta_{l-1}, \eta_l) \,\big|\, \mathcal{F}_t^{W,\mu}\Big),$$

where $\mathcal{V}_n = \{\nu \in \mathcal{V},\ \text{s.t. } \nu \in]0; n]\}$.

Theorem 2: sketch of proof (2/3)

2.b For all ν in $\mathcal{V}_n$, & taking $\mathbb{E}^{\nu}(|\mathcal{F}_t^{W,\mu})$

$$Y_t^n =$$

$$\mathbb{E}^{\nu}\Big(g(X, I_T) + \int_t^T f_s(X, I_s)\, ds - \sum_{\substack{l\geq 1 \\ t\leq\sigma_l<T}} c_{\sigma_l}(X, \eta_{l-1}, \eta_l) |\mathcal{F}_t^{W,\mu}\Big)$$

$$+\mathbb{E}^{\nu}\Big(\underbrace{\int_t^T \int_A \{n(\hat{U}_s^n(a))^+ - \hat{U}_s^n(a)\nu_s(a)\}\,\lambda(da)\, ds}_{\geq 0} |\mathcal{F}_t^{W,\mu}\Big)$$

with $\hat{U}_s^n(a) = U_s^n(a) - c_s(X, I_{s-}, a)$ & using $nx^+ - \nu x \geq 0$

Theorem 2: sketch of proof (2/3)

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2.b For all ν in $\mathcal{V}_n$, & taking $\mathbb{E}^\nu(|\mathcal{F}_t^{W,\mu})$

$$Y_t^n = \mathbb{E}^\nu\Big(g(X, I_T) + \int_t^T f_s(X, I_s)\, ds - \sum_{\substack{l \geq 1 \\ t \leq \sigma_l < T}} c_{\sigma_l}(X, \eta_{l-1}, \eta_l) \Big| \mathcal{F}_t^{W,\mu}\Big)$$

$$+\mathbb{E}^\nu\Big(\underbrace{\int_t^T \int_A \left\{ n(\hat{U}_s^n(a))^+ - \hat{U}_s^n(a)\nu_s(a) \right\} \lambda(da)\, ds}_{\geq 0} \Big| \mathcal{F}_t^{W,\mu}\Big)$$

with $\hat{U}_s^n(a) = U_s^n(a) - c_s(X, I_{s-}, a)$ & using $nx^+ - \nu x \geq 0$

2.c Setting $\nu^{\epsilon,n}(a) =$
$n 1_{\{\hat{U}_s^n(a) \geq 0\}} + \varepsilon\, 1_{\{-1 < \hat{U}_s^n(a) < 0\}} - \varepsilon(\hat{U}_s^n(a))^{-1}\, 1_{\{\hat{U}_s^n(a) \leq -1\}}$
then
$\nu^{\epsilon,n} \in]0, n]$ and $\nu^{\varepsilon,n} \leq \varepsilon$ on $\{\hat{U}_s^n(a) < 0\}$

Theorem 2: sketch of proof (3/3)

2.c $Y_t^n \leq \varepsilon(T-t)\lambda(A) +$

$$\mathbb{E}^{\nu^{\varepsilon,n}}\Big[\int_t^T f_s(X, I_s)ds + g(X, I_T) - \sum_{\substack{n\geq 1\\ t\leq\sigma_l<T}} c_{\sigma_n}(X, \eta_{n-1}, \eta_n)\Big|\mathcal{F}_t^{W,\mu}\Big]$$

Theorem 2: sketch of proof (3/3)

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3. By comparison (Royer '05), Y^n non decreasing, we set $Y = \lim \nearrow Y_t^n$.
4. (2.a) gives $\sup_n \mathbb{E}(\sup_t |Y_t^n|^2) < \infty$ & standard BSDE estimates $\exists\, C > 0, \ \ \forall n \ \ |Y^n|_{S^2} + |Z^n|_{L^2} + |U^n|_{L^2} + |K^n|_{S^2} \leq C.$

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$$\mathbb{E}^{\nu^{\varepsilon,n}}\Big[\int_t^T f_s(X, I_s)ds + g(X, I_T) - \sum_{\substack{n\geq 1\\ t\leq\sigma_I<T}} c_{\sigma_n}(X, \eta_{n-1}, \eta_n)\Big|\mathcal{F}_t^{W,\mu}\Big]$$

3. By comparison (Royer '05), Y^n non decreasing, we set $Y = \lim \nearrow Y_t^n$.
4. (2.a) gives $\sup_n \mathbb{E}(\sup_t |Y_t^n|^2) < \infty$ & standard BSDE estimates
   $\exists\, C > 0, \ \ \forall n \ \ |Y^n|_{S^2} + |Z^n|_{L^2} + |U^n|_{L^2} + |K^n|_{S^2} \leq C.$
5. An (extended) Peng's monotonic limit theorem

   $\forall\, 1 \leq p < 2, \quad |Y^n - Y|_{L^p} + |Z^n - Z|_{L^p} + |U^n - U|_{L^p} \to 0,$ and

   $K^n - K \xrightarrow{w} 0$ yielding existence.

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Thanks for your attention & happy birthday Ying.