Multidimensional BSDEs with rough drifts

Jiahao Liang   Shanjian Tang

Fudan University

19 June, 2024

at the conference A Backward Stochastic Excursion with Ying Hu on the Occation of his 60th Birthday, June 17-19, 2024, Université de Rennes, Rennes, France

Outline

1. Forwards
2. Rough BSDEs and related results
3. Case for g deterministic and time-invariant
4. Case for g linear

Outline

Forwards

Two Anniversaries of 60 Years

1. On January 27, China and France decided to establish their diplomatic relationship.

2. On April 10, Ying Hu was born in Jiangsu Province, China.

Last year saw two anniversaries of 50 years for ......and ......?

Happy Birthday Ying!

Outline

Rough BSDEs

A **rough BSDE** reads

$$Y_t = \xi + \int_t^T f_r(Y_r, Z_r)\, dr + \int_t^T g_r(Y_r)\, d\mathbf{X}_r - \int_t^T Z_r dW_r, \quad t \in [0, T].$$

- unknown pair of processes $(Y, Z) : \Omega \times [0, T] \to \mathbb{R}^k \times \mathbb{R}^{k \times d}$
- terminal value ξ: an $\mathbb{R}^k$-valued random variable
- coefficients f and g: progressively measurable vector fields
- W: a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$
- $\mathbf{X} = (X, \mathbb{X})$: an $\mathbb{R}^e$-valued p-rough path, $p \in [2, 3)$

$\mathbb{R}^e$-valued p-rough path $\mathbf{X} = (X, \mathbb{X})$

The space of continuous paths of finite p-variation is denoted by $C^{p\text{-}var}([0,T],V)$.

Definition

For $p \in [2,3)$, we call $\mathbf{X} = (X, \mathbb{X})$ a two-step p-rough path with values in $\mathbb{R}^e$, denoted by $\mathbf{X} \in \mathscr{C}^{p\text{-}var}([0,T],\mathbb{R}^e)$, if the following are satisfied

(i) $X \in C^{p\text{-}var}([0,T],\mathbb{R}^e)$;

(ii) $\mathbb{X} : \Delta \to \mathbb{R}^e \otimes \mathbb{R}^e$ is continuous and $|\mathbb{X}|_{\frac{p}{2}\text{-}var}$ is finite;

(iii) $\mathbf{X}$ satisfies Chen's relation

$$\delta\mathbb{X}_{s,u,t} = \delta X_{s,u} \otimes \delta X_{u,t}, \quad \forall (s,u,t) \in \Delta_2. \tag{1}$$

Existing results

- Diehl & Friz (AP, 2012): well-posedness of rough BSDEs for $k = 1$ with a flow transformation.
- Crisan, Diehl, Friz & Oberhauser (AAP, 2013): well-posedness of rough SDEs with the same flow transformation.
- Diehl, Oberhauser & Riedel (SPA, 2015): well-posedness of rough SDEs with a joint lift of $(W(\omega), \mathbf{X})$.
- Friz, Hocquet & Lê (2021): well-posedness of rough SDEs with a fixed-point argument.

Outline

Case for g deterministic and time-invariant

Consider the **rough BSDE** with deterministic and time-invariant coefficient g:

$$Y_t = \xi + \int_t^T f_r\left(Y_r, Z_r\right) dr + \int_t^T g\left(Y_r\right) d\mathbf{X}_r - \int_t^T Z_r dW_r. \tag{2}$$

We call $(Y, Z) \in \mathbb{S}^\infty \times BMO$ a **solution** to (2) if there exist $X^n \in C^\infty\left([0,T], \mathbb{R}^e\right)$ and $(Y^n, Z^n) \in \mathbb{S}^\infty \times BMO$ for $n = 1, 2, \cdots$, such that (Y^n, Z^n) is a solution to BSDE

$$Y_t^n = \xi + \int_t^T \left(f_r\left(Y_r^n, Z_r^n\right) + g\left(Y_r^n\right) \dot{X}_r^n\right) dr - \int_t^T Z_r^n dW_r,$$

and

$$\lim_{n\to\infty} \left(\rho_{p\text{-}var}\left(\mathbf{X}^n, \mathbf{X}\right) + \|Y^n - Y\|_\infty + \|Z^n - Z\|_{BMO}\right) = 0.$$

Assumptions and the main theorem

- $\mathbf{X}$ is a geometric p-rough path.
- $\xi \in L^\infty\left(\Omega, \mathcal{F}_T; \mathbb{R}^k\right)$.
- There exist $L \geq 0$ and measurable adapted processes $\lambda, \mu : \Omega \times [0,T] \to [0,\infty)$ such that $\left\| \int_0^T \left(\lambda_r + \mu_r^2\right) dr \right\|_\infty < \infty$,
$$|f_t(y,z)| + |\partial_y f_t(y,z)| \leq \lambda_t + L\left(|y|^2 + |z|^2\right),$$
$$|\partial_z f_t(y,z)| \leq \mu_t + L\left(|y| + |z|\right).$$
- $g \in Lip^\gamma\left(\mathbb{R}^k, \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)$ for $\gamma > p+2$.

Assumptions and the main theorem

- $\mathbf{X}$ is a geometric p-rough path.
- $\xi \in L^\infty\left(\Omega, \mathcal{F}_T; \mathbb{R}^k\right)$.
- There exist $L \geq 0$ and measurable adapted processes $\lambda, \mu : \Omega \times [0,T] \to [0,\infty)$ such that $\left\| \int_0^T \left(\lambda_r + \mu_r^2\right) dr \right\|_\infty < \infty$,

$$|f_t(y,z)| + |\partial_y f_t(y,z)| \leq \lambda_t + L\left(|y|^2 + |z|^2\right),$$

$$|\partial_z f_t(y,z)| \leq \mu_t + L\left(|y| + |z|\right).$$

- $g \in Lip^\gamma\left(\mathbb{R}^k, \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)$ for $\gamma > p+2$.

Theorem (Liang–Tang'23^+)

For $\|\xi\|_\infty$, $\left\| \int_0^T \left(\lambda_r + \mu_r^2\right) dr \right\|_\infty$ and $|\mathbf{X}|_{p\text{-}var}$ sufficiently small, the rough BSDE (2) has a unique solution (Y,Z) satisfying

$$\|Y\|_\infty + \|Z\|_{BMO} \lesssim \|\xi\|_\infty + \left\| \int_0^T \left(\lambda_r + \mu_r^2\right) dr \right\|_\infty + |\mathbf{X}|_{p\text{-}var}.$$

Flow transformation

Define

- $\phi : [0,T] \times \mathbb{R}^k \to \mathbb{R}^k$ be the solution flow to RDE

$$\phi_t(y) = y + \int_t^T g(\phi_r(y))\, d\mathbf{X}_r,$$

  and $\psi_t(\cdot)$ be the inverse of $\phi_t(\cdot)$,
- $\tilde{Y} := \psi(Y)$ and $\tilde{Z} := D\psi(Y)\, Z$,
- $\tilde{f} : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ by

$$\tilde{f}_t(\tilde{y}, \tilde{z}) := (D\phi_t(\tilde{y}))^{-1} \left(f_t(\phi_t(\tilde{y}), D\phi_t(\tilde{y})\, \tilde{z}) + \frac{1}{2} D^2\phi_t(\tilde{y})\, \tilde{z}^2 \right),$$

Quadratic BSDEs

Then the rough BSDE (2) is transformed into the BSDE:

$$\tilde{Y}_t = \xi + \int_t^T \tilde{f}_r\left(\tilde{Y}_r, \tilde{Z}_r\right) dr - \int_t^T \tilde{Z}_r dW_r, \quad t \in [0,T].$$

By Assumptions, we get estimates on $\tilde{f}$:

$$\left|\tilde{f}_t(\tilde{y}, \tilde{z})\right| + \left|\partial_{\tilde{y}} \tilde{f}_t(\tilde{y}, \tilde{z})\right| \lesssim \lambda_t + \mu_t^2 + |\mathbf{X}|_{p\text{-}var} + |\tilde{y}|^2 + |\tilde{z}|^2,$$

$$\left|\partial_{\tilde{z}} \tilde{f}_t(\tilde{y}, \tilde{z})\right| \lesssim \mu_t + |\mathbf{X}|_{p\text{-}var} + |\tilde{y}| + |\tilde{z}|.$$

Tevzadze (SPA, 2008): multidimensional quadratic BSDEs are well-posed for small terminal value.

Outline

Some notations of space

For $q \in [1,\infty)$ and $m \in [1,\infty]$ define

- $\Delta := \{(s,t) : 0 \le s \le t \le T\}$,
- $C_2^{q\text{-}var} L_m\left([0,T], \Omega; V\right)$ as the space of measurable adapted processes $A : \Omega \times \Delta \to V$ such that $A \in C\left(\Delta, L^m\left(\Omega; V\right)\right)$ and

$$\|A\|_{m,q\text{-}var} := \sup_{\pi \in \mathcal{P}([0,T])} \left(\sum_{[u,v]\in\pi} \|A_{u,v}\|_m^q \right)^{\frac{1}{q}} < \infty,$$

- $C^{q\text{-}var} L_m\left([0,T], \Omega; V\right)$ as the space of measurable adapted processes $Y : \Omega \times [0,T] \to V$ such that $Y \in C\left([0,T], L^m\left(\Omega; V\right)\right)$ and $\|\delta Y\|_{m,q\text{-}var} < \infty$, where $\delta Y_{s,t} := Y_t - Y_s$, endowed with the norm

$$\|Y\|_{C^{q\text{-}var} L_m} := \|Y_T\|_m + \|\delta Y\|_{m,q\text{-}var} .$$

Stochastic controlled rough paths

We call (Y, Y') an L^m-integrable V-valued **stochastic controlled rough path**, denoted by $(Y, Y') \in \mathbf{D}_X^{(p,p)\text{-}var} L_m([0,T], \Omega; V)$, if

- $Y \in C^{p\text{-}var} L_m([0,T], \Omega; V)$,
- $Y' \in C^{p\text{-}var} L_m([0,T], \Omega; \mathcal{L}(\mathbb{R}^e, V))$,
- $\mathbb{E}.R^Y \in C_2^{(p/2)\text{-}var} L_m([0,T], \Omega; V)$, where $R^Y_{s,t} := \delta Y_{s,t} - Y'_s \delta X_{s,t}$.

We call (Y, Y') a **controlled rough path** with values in $L^m(\Omega; V)$, denoted by $(Y, Y') \in \mathscr{D}_X^{(p,p)\text{-}var} L_m([0,T], \Omega; V)$ if additionally

- $R^Y \in C_2^{(p/2)\text{-}var} L_m$.

Decomposition

Decomposition lemma (Friz–Hocquet–L'21^+)

For any $(Y, Y') \in \mathbf{D}_X^{(p,p)\text{-}var} L_m$ with $m \in [2, \infty)$, there exists a unique pair of processes (Y^M, Y^J) such that $Y^M \in C^{p\text{-}var} L_m$ is an L^m-integrable martingale with $Y_0^M = 0$, $(Y^J, Y') \in \mathscr{D}_X^{(p,p)\text{-}var} L_m$ and $Y = Y^M + Y^J$.

Decomposition

Decomposition lemma (Friz–Hocquet–L'21^+)

For any $(Y, Y') \in \mathbf{D}_X^{(p,p)\text{-}var} L_m$ with $m \in [2, \infty)$, there exists a unique pair of processes (Y^M, Y^J) such that $Y^M \in C^{p\text{-}var} L_m$ is an L^m-integrable martingale with $Y_0^M = 0$, $(Y^J, Y') \in \mathscr{D}_X^{(p,p)\text{-}var} L_m$ and $Y = Y^M + Y^J$.

For $K \geq 1$ define

$$\left\| (Y, Y') \right\|^{(K)}_{\mathbf{D}_X^{(p,p)\text{-}var} L_m} := \| Y_T \|_m + \left\| Y'_T \right\|_m + K \left\| \delta Y^M \right\|_{m,p\text{-}var} + \left\| \delta Y' \right\|_{m,p\text{-}var} + K \left\| R^{Y^J} \right\|_{m,(p/2)\text{-}var}$$

Sewing lemma

Let $A : \Omega \times \Delta \to V$ be a measurable adapted L^m-integrable process. Assume that there exist positive constants ε_i and controls w_i for $1 \le i \le N$ such that

$$\|\delta A_{s,u,t}\|_m \le \sum_{i=1}^{N} w_i\,(s,t)^{1+\varepsilon_i}$$

where $\delta A_{s,u,t} = A_{s,t} - A_{s,u} - A_{u,t}$. Then there exists a unique measurable adapted L^m-integrable process $\mathcal{A} : \Omega \times [0,T] \to V$ with $\mathcal{A}_0 = 0$ such that

$$\|\delta \mathcal{A}_{s,t} - A_{s,t}\|_m \lesssim \sum_{i=1}^{N} w_i\,(s,t)^{1+\varepsilon_i}.$$

Moreover, we have

$$\lim_{\pi \in \mathcal{P}([0,T]), |\pi| \to 0} \sup_{t \in [0,T]} \left\| \mathcal{A}_t - \sum_{[u,v] \in \pi, u \le t} A_{u,v \wedge t} \right\|_m = 0.$$

Stochastic sewing lemma

Let $A : \Omega \times \Delta \to V$ be a measurable adapted L^m-integrable process. Assume that there exist positive constants ε_i and controls w_i for $1 \leq i \leq N$ such that

$$\|\delta A_{s,u,t}\|_m \leq \sum_{i=1}^{N} w_i\,(s,t)^{\frac{1}{2}+\varepsilon_i}\,, \quad \mathbb{E}_s \delta A_{s,u,t} = 0,$$

where $\delta A_{s,u,t} = A_{s,t} - A_{s,u} - A_{u,t}$. Then there exists a unique measurable adapted L^m-integrable process $\mathcal{A} : \Omega \times [0,T] \to V$ with $\mathcal{A}_0 = 0$ such that

$$\|\delta \mathcal{A}_{s,t} - A_{s,t}\|_m \lesssim \sum_{i=1}^{N} w_i\,(s,t)^{\frac{1}{2}+\varepsilon_i}\,, \quad \mathbb{E}_s\left[\delta \mathcal{A}_{s,t} - A_{s,t}\right] = 0.$$

Moreover, we have

$$\lim_{\pi \in \mathcal{P}([0,T]), |\pi| \to 0} \sup_{t \in [0,T]} \left\| \mathcal{A}_t - \sum_{[u,v] \in \pi, u \leq t} A_{u, v \wedge t} \right\|_m = 0.$$

p-rough stochastic integration

Let $(Y, Y') \in \mathbf{D}_X^{(p,p)\text{-}var} L_m \left([0,T], \Omega; \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)$. Then there exists a unique measurable adapted L^m-integrable process $\int_0^\cdot Y \, d\mathbf{X} : \Omega \times [0,T] \to \mathbb{R}^k$ with the vanishing initial value such that

$$\lim_{\pi \in \mathcal{P}([0,T]), |\pi| \to 0} \sup_{t \in [0,T]} \left\| \int_0^t Y \, d\mathbf{X} - \sum_{[u,v] \in \pi, u \leq t} \left(Y_u \delta X_{u, v\wedge t} + Y'_u \mathbb{X}_{u, v \wedge t} \right) \right\|_m = 0.$$

We call $\int_0^\cdot Y \, d\mathbf{X}$ the p-**rough stochastic integral** of (Y, Y') against $\mathbf{X}$. Moreover, $\left(\int_0^\cdot Y \, d\mathbf{X}, Y\right) \in \mathscr{D}_X^{(p,p)\text{-}var} L_m \left([0,T], \Omega; \mathbb{R}^k\right)$ and for any $K \geq 1$,

$$\begin{aligned} &\left\| \left(\int_0^\cdot Y \, d\mathbf{X}, Y \right) \right\|^{(K)}_{\mathbf{D}_X^{(p,q)\text{-}var} L_m} \\ &\quad \leq \|Y_T\|_m + C \left\| (Y, Y') \right\|^{(K)}_{\mathbf{D}_X^{(q,q')\text{-}var} L_m} \left(\frac{1}{K} + K \, |\mathbf{X}|_{p\text{-}var} \right). \end{aligned}$$

Proof ideas

- Define $A^M_{s,t} := Y^M_s \delta X_{s,t}$ and $A^J_{s,t} := Y^J_s \delta X_{s,t} + Y'_s \mathbb{X}_{s,t}$.
- Apply stochastic sewing lemma to A^M to obtain $\int_0^\cdot Y^M d\mathbf{X}$.
- Apply sewing lemma to A^J to obtain $\int_0^\cdot Y^J d\mathbf{X}$.
- Define $\int_0^\cdot Y d\mathbf{X} := \int_0^\cdot Y^M d\mathbf{X} + \int_0^\cdot Y^J d\mathbf{X}$.

Consider the **BSDE with a linear rough drift**:

$$Y_t = \xi + \int_t^T f_r(Y_r, Z_r)\, dr + \int_t^T (G_r Y_r + H_r)\, d\mathbf{X}_r - \int_t^T Z_r dW_r. \quad (3)$$

We call $(Y, Z) \in \mathbb{L}^2 \times \mathbb{L}^2$ a solution to (3) if

- $\int_0^T |f_r(Y_r, Z_r)|\, dr$ is finite a.s.,
- $\left(GY + H, (GY + H)'\right) \in \mathbf{D}_X^{(p,p)\text{-}var} L_2\left([0, T], \Omega; \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)$, where $(GY + H)' := G(GY + H) + G'Y + H'$,
- for every $t \in [0, T]$,

$$Y_t = \xi + \int_t^T f_r(Y_r, Z_r)\, dr + \int_t^T (GY + H)\, d\mathbf{X} - \int_t^T Z_r dW_r \quad a.s.$$

Assumptions and the main theorem

Assumptions:

- $\xi \in L^2\left(\Omega, \mathcal{F}_T; \mathbb{R}^k\right)$ and $f(0,0) \in \mathbb{L}^2\left([0,T], \Omega; \mathbb{R}^k\right)$.
- f is uniformly Lipschitz continuous in y and z.
- $(G, G') \in \mathbf{D}_X^{(p,p)\text{-}var} L_\infty\left([0,T], \Omega; \mathcal{L}\left(\mathbb{R}^k, \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)\right)$.
- $(H, H') \in \mathbf{D}_X^{(p,p)\text{-}var} L_2\left([0,T], \Omega; \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)$.

Assumptions and the main theorem

Assumptions:

- $\xi \in L^2\left(\Omega, \mathcal{F}_T; \mathbb{R}^k\right)$ and $f(0,0) \in \mathbb{L}^2\left([0,T], \Omega; \mathbb{R}^k\right)$.
- f is uniformly Lipschitz continuous in y and z.
- $(G, G') \in \mathbf{D}_X^{(p,p)\text{-}var} L_\infty\left([0,T], \Omega; \mathcal{L}\left(\mathbb{R}^k, \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)\right)$.
- $(H, H') \in \mathbf{D}_X^{(p,p)\text{-}var} L_2\left([0,T], \Omega; \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)$.

Theorem (Liang–Tang'23^+)

Under the above Assumptions, the rough BSDE (3) has a unique solution.

Assumptions and the main theorem

Assumptions:

- $\xi \in L^2\left(\Omega, \mathcal{F}_T; \mathbb{R}^k\right)$ and $f(0,0) \in \mathbb{L}^2\left([0,T], \Omega; \mathbb{R}^k\right)$.
- f is uniformly Lipschitz continuous in y and z.
- $(G, G') \in \mathbf{D}_X^{(p,p)\text{-}var} L_\infty\left([0,T], \Omega; \mathcal{L}\left(\mathbb{R}^k, \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)\right)$.
- $(H, H') \in \mathbf{D}_X^{(p,p)\text{-}var} L_2\left([0,T], \Omega; \mathcal{L}\left(\mathbb{R}^e, \mathbb{R}^k\right)\right)$.

Theorem (Liang–Tang'23^+)

Under the above Assumptions, the rough BSDE (3) has a unique solution.

Auxiliary results:

- $F \in \mathbb{L}^2 \Rightarrow \int_0^\cdot F_r dr \in C^{1\text{-}var} L_2 \Rightarrow \left(\int_0^\cdot F_r dr, 0\right) \in \mathscr{D}_X^{(p,p)\text{-}var} L_2$.
- M is a square-integrable martingale $\Rightarrow M \in C^{2\text{-}var} L_2$.
- $(Y, Y') \in \mathbf{D}_X^{(p,p)\text{-}var} L_2 \Rightarrow (GY, GY' + G'Y) \in \mathbf{D}_X^{(p,p)\text{-}var} L_2$.

Fixed point argument

Define

$$\mathbf{S} := \left\{ \left(Y, Y', Z\right) \in \mathbf{D}_X^{(p,p)\text{-}var} L_2 \times \mathbb{L}^2 : Y_T = \xi, Y'_T = G_T \xi + H_T \right\},$$

$$\left\| \left(Y, Y', Z\right) \right\|^{(K)} := \left\| \left(Y, Y'\right) \right\|^{(K)}_{\mathbf{D}_X^{(p,p)\text{-}var} L_2} + K \left\| Z \right\|_2 .$$

For any $(Y, Y', Z) \in \mathbf{S}$, $M : \Omega \times [0, T] \to \mathbb{R}^k$ defined by

$$M_t := \mathbb{E}_t \left[\xi + \int_0^T f_r \left(Y_r, Z_r\right) dr + \int_0^T \left(GY + H\right) d\mathbf{X} \right]$$

is a square-integrable martingale and there exists unique Φ^Z such that

$$M_t = M_0 + \int_0^t \Phi_r^Z dW_r.$$

Define $\Phi^{Y'} := GY + H$ and

$$\Phi^Y := \xi + \int_\cdot^T f_r \left(Y_r, Z_r\right) dr + \int_\cdot^T \left(GY + H\right) d\mathbf{X} - \int_\cdot^T \Phi_r^Z dW_r.$$

Thank you for your attention!

Ying,
Happy Birthday!
More Happiness, More Luckies and More Successes
in the future!