Logarithmic energy for zeros of random polynomials

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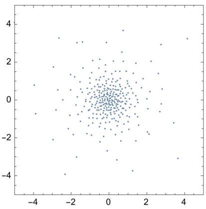
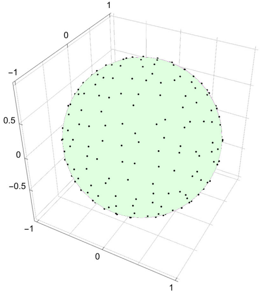




Random nodal domains, Rennes, June 2023

Let $x_1, \ldots, x_n \in \mathbb{S}^2 \subset \mathbb{R}^3$. The logarithmic energy is given by

$$\mathcal{E}_n(x_1, \ldots, x_n) = \sum_{i \neq j} \log \frac{1}{\|x_i - x_j\|}.$$

Stephan Smale's 7th problem:

Problem 7: Distribution of Points on the 2-Sphere

Let $V_N(x) = \sum_{1 \leq i < j \leq N} \log \frac{1}{\|x_i - x_j\|}$, where $x = (x_1, \ldots, x_N)$, the x_i are distinct points on the 2-sphere $S^2 \subset \mathbb{R}^3$, and $\|x_i - x_j\|$ is the distance in $\mathbb{R}^3$. Denote $\min_x V_N(x)$ by V_N.

Find $(x_1, \ldots, x_N)$ *such that*

$$V_N(x) - V_N \leq c \log N, \quad c \text{ a universal constant.} \qquad (2)$$

"Mathematical problems for the next century" by S. Smale (1998)

"Find" - an algorithm with polynomial running time.

$$m_n = \min_{x_1,\ldots,x_n \in \mathbb{S}^2} \sum_{i \neq j} \log \frac{1}{\|x_i - x_j\|}$$

How does m_n behave as $n \to \infty$?

- Fekete (1923): $m_n = \left(\frac{1}{2} - \log 2\right)n^2 + o(n^2)$
- Wagner (1989): $m_n = \left(\frac{1}{2} - \log 2\right)n^2 - \frac{1}{2}n\log n + O(n)$
- Bétermin-Sandier (2018): $\exists C_{\mathsf{min}}$ so that

$$m_n = \left(\frac{1}{2} - \log 2\right)n^2 - \frac{1}{2}n\log n + C_{\mathsf{min}}n + o(n)$$

- It is conjectured (Bétermin-Sandier, Brauchart-Hardin-Saff)

$$C_{\mathsf{min}} = 2\log 2 + \frac{1}{2}\log\frac{2}{3} + 3\log\frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.0566\ldots$$

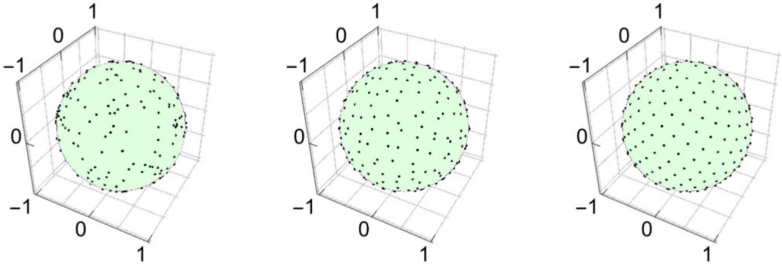
- Equivalent to the Cohn-Cumar conjecture in $d = 2$!

$$\mathcal{E}_n(\mathbf{x}) = -\sum_{i \neq j} \log \|x_i - x_j\| \qquad m_n = \min_{\mathbf{x}} \mathcal{E}_n(\mathbf{x})$$

- Natural idea: study random sets on $\mathbb{S}^2$ with small log-energy.
- Put n independent uniformly distributed points on $\mathbb{S}^2$

$$\mathbb{E}_{\mathsf{Uniform}}[\mathcal{E}_n] = \Big(\frac{1}{2} - \log 2\Big)n^2 - \Big(\frac{1}{2} - \log 2\Big)n$$

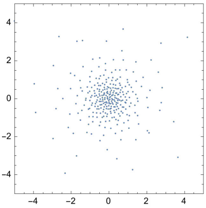
- Agrees with m_n up to a $\Theta(n \log n)$ term;
- Fluctuations are of typically of order n.

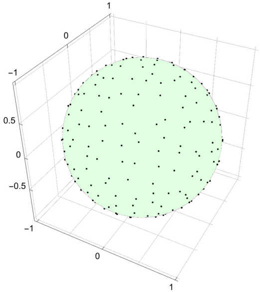


Kostlan-Shub-Smale model for random polynomials:

$$f_n(z) = \sum_{k=0}^{n} a_k \sqrt{\binom{n}{k}} z^k \qquad z \in \mathbb{C}$$

where $\{a_k\}$ are i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$.





Zero set has distribution invariant under rotations!

Kostlan-Shub-Smale model for random polynomials:

$$f_n(z) = \sum_{k=0}^{n} a_k \sqrt{\binom{n}{k}} z^k \qquad a_k \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ i.i.d.}$$

Let $\{\zeta_1, \ldots, \zeta_n\}$ be the roots of f_n on $\mathbb{S}^2$. Set

$$\mathcal{E}_n(\zeta_1, \ldots, \zeta_n) = -\sum_{i \neq j} \log \|\zeta_i - \zeta_j\|$$

Theorem (Armentano-Beltrán-Shub 2011)

Let $\{\zeta_1, \ldots, \zeta_n\}$ be the random zeros of f_n, then

$$\mathbb{E}\big[\mathcal{E}_n(\zeta_1, \ldots, \zeta_n)\big] = \Big(\frac{1}{2} - \log 2\Big) n^2 - \frac{1}{2} n \log n - \Big(\frac{1}{2} - \log 2\Big) n$$

Agrees with the minimum m_n up to a $\Theta(n)$ term

Kostlan-Shub-Smale model for random polynomials:

$$f_n(z) = \sum_{k=0}^{n} a_k \sqrt{\binom{n}{k}} z^k \qquad a_k \sim \mathcal{N}_{\mathbb{C}}(0,1) \text{ i.i.d.}$$

$\mathcal{E}_n$ - logarithmic energy of the roots; Armentano-Beltrán-Shub:

$$\mathbb{E}[\mathcal{E}_n] = \Big(\frac{1}{2} - \log 2\Big)n^2 - \frac{1}{2} n \log n - \Big(\frac{1}{2} - \log 2\Big)n$$

Theorem (Michelen-Y. 2023)

$\exists c_* > 0$ *such that* $\mathrm{Var}[\mathcal{E}_n] \sim c_* n$ *and furthermore*

$$\frac{\mathcal{E}_n - \mathbb{E}[\mathcal{E}_n]}{\sqrt{c_* n}} \xrightarrow[n\to\infty]{d} \mathcal{N}_{\mathbb{R}}(0,1)\,.$$

- $c_* \approx 0.0907...$ is explicit (but ugly)
- Proof implies concentration of all moments at the scale of $\sqrt{n}$

$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere; the induced metric

$$d_{\mathbb{S}^2}(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \qquad z, w \in \widehat{\mathbb{C}}$$

The invariant measure

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}m(z)}{\pi(1+|z|^2)^2}\,, \qquad \int_{\mathbb{C}} \mathrm{d}\mu(z) = 1\,.$$

Kostlan-Shub-Smale model for random polynomials:

$$f_n(z) = \frac{1}{(1+|z|^2)^{n/2}} \sum_{k=0}^{n} a_k \sqrt{\binom{n}{k}} z^k \qquad a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

The logarithmic energy of the zeros $\{\zeta_1, \ldots, \zeta_n\}$ of f_n:

$$\mathcal{E}_n = -\sum_{i \neq j} \log d_{\mathbb{S}^2}(\zeta_i, \zeta_j)\,.$$

$$f_n(z) = \frac{1}{(1+|z|^2)^{n/2}} \sum_{k=0}^{n} a_k \sqrt{\binom{n}{k}} z^k \qquad a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

- f_n is a complex Gaussian process on $\widehat{\mathbb{C}}$, with covariance

$$K_n(z,w) = \mathbb{E}\Big[f_n(z)\overline{f_n(w)}\Big] = \left(\frac{1+z\overline{w}}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}\right)^n$$

- $|K_n(z,w)|^2 = \Big(1 - d_{\mathbb{S}^2}(z,w)^2/4\Big)^n$
- Therefore, f_n starts to decorrelate at scale $\asymp n^{-1/2}$.
- Gaussian Entire Function:

$$g(z) = e^{-|z|^2/2} \sum_{k=0}^{\infty} a_k \frac{z^k}{\sqrt{k!}} \qquad z \in \mathbb{C}$$

- Key observation: $\lim_{n\to\infty} f_n\left(\frac{z}{\sqrt{n}}\right) = g(z)$ $\left(\text{since } \binom{n}{k} \sim \frac{n^k}{k!}\right)$

Lemma: Let $\mathcal{E}_n$ be the log-energy of f_n, then

$$\mathcal{E}_n = \Big(\frac{1}{2} - \log 2\Big)n^2 - \frac{n \log n}{2} + (\log 2)n + \mathcal{I}_n - \mathcal{S}_n$$

$$\mathcal{I}_n = n \int_{\mathbb{C}} \log |f_n(z)| \, \mathrm{d}\mu(z) \,, \quad \mathcal{S}_n = \sum_{\zeta : f_n(\zeta)=0} \log \Big| \frac{f_n'(\zeta)}{\sqrt{n}} (1+|\zeta|^2) \Big| .$$

- $\mathcal{I}_n$ and $\mathcal{S}_n$ are rotation invariant. Note that

$$\frac{f_n'(z)}{\sqrt{n}} (1 + |z|^2) \Big|_{\{f_n(z)=0\}} \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

- Simple computation: $\mathbb{E}[\mathcal{I}_n] = -\frac{\gamma}{2} n$ and $\mathbb{E}[\mathcal{S}_n] = \frac{1-\gamma}{2} n$
- Our Goal: $n^{-1/2}\big(\mathcal{I}_n, \mathcal{S}_n\big) \xrightarrow[n\to\infty]{d}$ 2D Gaussian law
- Goal is achieved by computing the joint moments of $\mathcal{I}_n$ and $\mathcal{S}_n$.

$$f_n(z) = \frac{1}{(1+|z|^2)^{n/2}} \sum_{k=0}^{n} a_k \sqrt{\binom{n}{k}} z^k \qquad a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

ρ_k^n - k-point function of the zero set. Kac-Rice formulas:

$$\rho_1^n(z) = \frac{1}{\pi}\mathbb{E}\big[|f_n'(z)|^2 \mid f_n(z)=0\big] = \frac{n}{\pi(1+|z|^2)^2}$$

$$\rho_2^n(z,w) = \frac{\mathbb{E}\big[|f_n'(z)|^2|f_n'(w)|^2 \mid f_n(z)=f_n(w)=0\big]}{\pi^2 \det \mathsf{Cov}\big(f_n(z), f_n(w)\big)}$$

Theorem (Michelen-Y. 2023, Clustering property)

For any $Z=(Z_1,Z_2)\in\mathbb{C}^j\times\mathbb{C}^{k-j}$ *we have*

$$\big|\rho_k^n(Z) - \rho_j^n(Z_1)\rho_{k-j}^n(Z_2)\big| \le C_k\, n^k \exp\Big(-c_k n\, d_{\mathbb{S}^2}(Z_1,Z_2)^2\Big)$$

- ◦ Clustering of GEF zeros ($n=\infty$) Nazarov-Sodin (2012)
- ◦ Similar result in $\mathbb{R}$: Ancona-Letendre (2020) Gass (2021)

Thank you! ☺