Graphon mean-field BSDEs with jumps and associated dynamic risk measures

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- Motivated by various applications, mean-field systems and mean-field games on large networks have been explored for different random graph models, e.g. Erdös-Rényi graph (Delarue 2017) or heterogeneous random graphs (Oliveira and Reis 2019).
- Recently, the use of graphons has emerged in order to analyze heterogeneous interactions in mean-field systems and game theory, (see Caines, Carmona, Bayraktar, Lacker, Laurière ...)
- Our goal here is to study Graphon Mean Field BSDEs with jumps

Outline

1. **BSDE with jumps with general mean-field operators**
   The driver of the BSDE contains a mean-field term which can accomodate several types of interactions ; in particular higher order interactions
2. **Graphon Mean Field BSDEs**
   Extend the study by the introduction of graphon interaction in the driver to capture heterogeneous interactions

Mean-field BSDEs with jumps

Let $(\Omega, \mathcal{F}, P)$ be a probability space ; Let W be a Brownian motion ; $\tilde{N}(dt, de)$ the compensated process of a Poisson r.m. $N(dt, de)$ with compensator $\nu(de)dt$ s.t. ν is a σ-finite positive measure on $\mathbb{R}^*$.
Let $\boldsymbol{F} = \{\mathcal{F}_t, t \geq 0\}$ the natural filtration associated with W and N. Let $T > 0$.

$$-dX_t = f(t, \omega, F(t, X_t), X_t, Z_t, l_t(\cdot))dt - Z_t dW_t - \int_{\mathbb{R}^*} l_t(e)\tilde{N}(dt, de)$$
$$X_T = \xi \in L^2(\mathcal{F}_T)$$

where f is a *Lipschitz driver* and F is a Lipschitz mean-field operator $F : [0, T] \times L^2(\mathcal{F}_T) \to \mathbb{R}$, $(t, X) \mapsto F(t, X)$ is measurable, $\forall t \in [0, T]$, $F(t, 0) < +\infty$, and $\exists\ C \geq 0$ s.t. $\forall (X_1, X_2) \in L^2(\mathcal{F}_T) \times L^2(\mathcal{F}_T)$, $|F(t, X_1) - F(t, X_2)| \leq C\|X_1 - X_2\|_2$.

Examples of mean-field operators

- First order interactions : $F(t,X) := \mathbb{E}[\phi(t,X)]$, where ϕ is a Lipschitz function s.t. $\phi(t,X) \in L^2(\mathcal{F}_T)$.

- Second order mean-field interaction term :

$$F(t,X) = \int_{\mathbb{R}\times\mathbb{R}} \kappa(x,x')\mu_t(dx)\mu_t(dx') = \mathbb{E}\left[\kappa(X,X')\right], ((X,X') \sim \mu_t \otimes \mu_t)$$

where κ is a Lipschitz kernel that captures the intensity of interactions, and X' is an independent copy of the same distribution μ_t as X.
The operator F may represent the average intensity of interactions of nodes in an inhomogeneous random graph (Bollobas et al 07).

$\mapsto$ When the kernel κ is constant in its first argument, we recover the expectation operator.

Results

- Existence and Uniqueness results
- Comparison theorems under appropriate monotony assumptions on f and F
- Dual representation in the convex case

R. Chen, R. Dumitrescu, A. Minca, and A.S. : Mean-field BSDEs with jumps and dual representation for global dynamic risk measures. *Probability, Uncertainty and Quantitative Risk*, 8(1) :33–52, 2023

We now turn to the study a class of mean field BSDEs with coupling specified via a graphon, to account for heterogeneity of a continuum of agents.

- Graphons have been developed by Lovász et al., as a natural continuum limit object for large dense graphs.
- A graphon is a symmetric measurable fn $G : I^2 \to I$, with $I := [0, 1]$ indexing a continuum of possible positions for nodes in the graph and $G(u, v)$ representing the edge density between nodes placed at u and v.

  The so-called cut norm of a graphon is defined by

$$\|G\|_{\square} := \sup_{A,B \in \mathcal{B}(I)} \left| \int_{A \times B} G(u, v) du dv \right|.$$

- We can also view a graphon as an operator from $L^{\infty}(I)$ to $L^1(I)$, defined for any $\phi \in L^{\infty}(I)$ as :

$$G\phi(u) := \int_I G(u, v)\phi(v) dv.$$

Spaces of processes

Let $\mathbb{J} = \mathcal{J}_t, t \geq 0$ be a filtration defined on some probability space

- $L^2(\mathcal{J}_t)$: set of all $\mathcal{J}_t$-measurable and square integrable r.v. for $t \in [0, T]$.
- $\mathbb{H}^2(\mathbb{J})$ is the set of all real-valued $\mathbb{J}$-predictable processes ϕ such that
$$\|\phi\|_{\mathbb{H}^2} := (\mathbb{E}[\int_0^T \phi_t^2 dt])^{1/2} < \infty.$$
- $\mathbb{H}^2_{\nu_u}(\mathbb{J})$ (for each $u \in I$) is the set of all $\mathbb{J}$-predictable function-valued processes ℓ such that
$$\|\ell\|_{\mathbb{H}^2_{\nu_u}} := (\mathbb{E}[\int_0^T \|\ell_t\|^2_{\nu_u} dt])^{1/2} < \infty.$$
- $\mathbb{S}^2(\mathbb{J})$ is the set of all real-valued RCLL $\mathbb{J}$-adapted processes ϕ with
$$\|\phi\|_{\mathbb{S}^2} := (\mathbb{E}[\sup_{t \in [0,T]} |\phi_t|^2])^{1/2} < \infty.$$
- $\mathcal{M}\mathbb{S}^2(\mathbb{J})$ is the set of all measurable functions X from I to $\mathbb{S}^2(\mathbb{J}) : u \mapsto X_u$, satisfying $\sup_{u \in I} \|X_u\|^2_{\mathbb{S}^2} = \sup_{u \in I} \mathbb{E}[\sup_{t \in [0,T]} |X_u(t)|^2] < \infty$.

Probability setup

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $I = [0,1]$.

- Let $\{W_u : u \in I\}$ be a family of independent Brownian motions.
- Let $\{N_u(dt,de) : u \in I\}$ be a family of independent Poisson measures with compensator $\nu_u(de)dt$ such that ν_u is a probability measure on $E := \mathbb{R}_*$, for each $u \in I$. Let $\{\widetilde{N}_u(dt,de) : u \in I\}$ be their compensator processes.
- Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with $\{W_u : u \in I\}$ and $\{N_u(dt,de) : u \in I\}$.
  Let $T > 0$. Denote by P the predictable σ-algebra on $[0,T] \times \Omega$.
- For each $u \in I$, let $\mathbb{F}^u = \{\mathcal{F}_t^u, t \geq 0\}$ be the augmented filtration generated by W_u and N_u.

Graphon mean-field BSDEs with jumps

$$
\begin{aligned}
X_u(t) =& \xi_u + \int_t^T \int_I \int_{\mathbb{R}} G(u,y) f(s,x,X_u(s^-),Z_u(s),\ell_{u,s}(\cdot))\mu_{y,s}(dx)dyds \\
&- \int_t^T Z_u(s)dW_u(s) - \int_t^T \int_E \ell_{u,s}(e)\widetilde{N}_u(ds,de), \quad \text{for } t \in [0,T],\ u \in I,
\end{aligned}
\tag{4.1}
$$

where $\mu_y := \mathcal{L}(X_y)$ and $\mu_{y,s} := \mathcal{L}(X_y(s))$. Assume $\forall u \in I$, $\xi_u \in L^2(\mathcal{F}_T^u)$.

Definition

A solution consists of a family of processes $\Phi := (X_u, Z_u, \ell_u)_{u \in I}$ with (X_u, Z_u, ℓ_u) in $\mathbb{S}^2(\mathbb{F}^u) \times \mathbb{H}^2(\mathbb{F}^u) \times \mathbb{H}^2_{\nu_u}(\mathbb{F}^u)$ for each u in I, satisfying (4.1), s.t. $u \mapsto \mathcal{L}(X_u)$ is measurable in the weak sense, X_u is a RCLL $\mathbb{R}$-valued optional process, and Z_u (resp. ℓ_u) is a $\mathbb{R}$-valued predictable process defined on $\Omega \times [0,T]$ (resp. $\Omega \times [0,T] \times E$) s.t. the stochastic integral is well defined.

Assumption on f : For each $u \in I$, we assume that $f : \Omega \times [0,T] \times \mathbb{R}^3 \times L^2_{\nu_u} \to \mathbb{R}$; $(\omega, t, x', x, z, \ell(\cdot)) \mapsto f(\omega, t, x', x, z, \ell(\cdot))$ is $P \otimes \mathcal{B}(\mathbb{R}^3) \otimes \mathcal{B}(L^2_{\nu_u})$ measurable, satisfies $f(\cdot,\cdot,0,0,0,0) \in \mathbb{H}^2(\mathbb{F}^u)$, and f is Lipschitz-continuous in (x', x, z, ℓ), i.e., $\exists$ a constant $C \geq 0$ s.t. $dt \otimes d\mathbb{P}$-a.s., for each (x'_1, x_1, z_1, ℓ_1) and (x'_2, x_2, z_2, ℓ_2), we have

$$
\begin{aligned}
&\left| f(\omega, t, x'_1, x_1, z_1, \ell_1(\cdot)) - f(\omega, t, x'_2, x_2, z_2, \ell_2(\cdot)) \right| \\
\leq & C(|x'_1 - x'_2| + |x_1 - x_2| + |z_1 - z_2| + \|\ell_1 - \ell_2\|_{\nu_u}).
\end{aligned}
$$

Graphon Mean-field BSDEs

Agenda

- existence, uniqueness and weak measurability of the solution
- comparison theorems under a monotonicity condition
- continuity of the solution with respect to the label index and stability of the system
- convergence of an interacting mean-field particle system with heterogeneous interactions to the graphon MFBSDE
- associated graphon dynamic risk measure and its properties
- dual representation in the convex case.

Canonical coupling

$$X_u(t) = \xi_u + \int_t^T \int_I \int_{\mathbb{R}} G(u,y) f(s,x,X_u(s^-),Z_u(s),\ell_{u,s}(\cdot))\mu_{y,s}(dx)dyds$$
$$- \int_t^T Z_u(s)dW_u(s) - \int_t^T \int_E \ell_{u,s}(e)\widetilde{N}_u(ds,de), \quad \text{for } t \in [0,T],\ u \in I,$$

- Note that different labels interact only through their laws μ_u.
- To handle the measurability of $\mathcal{L}(X_u)$ in u we can treat all processes $(X_u, Z_u, \ell_u)_{u \in I}$ on one stochastic basis : To this purpose, we introduce a canonical probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, on which we define a canonical Brownian motion $\bar{W}$ and a common Poisson random measure $\bar{N}(dt,de)$ with compensator $\nu(de)dt$ that is specified below. We define the canonical filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, where $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t, t \geq 0\}$ is the completed natural filtration and $\bar{\mathbb{P}}$ is the corresponding probability measure, generated by $\bar{W}$ and $\bar{N}(dt,de)$.

We transform the original graphon system into a fully coupled system driven by a common $(\bar{W},\bar{N})$, defined on a canonical space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{F}},\bar{\mathbb{P}})$, which admits a solution with the same law. We use trivially the canonical Brownian motion $\bar{W}$, and make the following assumption for the jump part :

Assumption :[Intensity measure]

- The function $I\times[1,2]\ni(u,w)\mapsto\varphi_u^{-1}(\omega-1)\in\mathbb{R}$ is $\mathcal{B}(I)\otimes\mathcal{B}([1,2])$ measurable, where φ_u denotes the cumulative distribution function of ν_u. We define $\varphi_u^{-1}(1)$ as the essential supremum and $\varphi_u^{-1}(0)$ as the essential infimum.

The idea is to use a common Poisson r.m. $\bar{N}$ to generate different r.m. N_u for all $u\in I$ through the mapping $\varphi_u^{-1}, u\in I$. Thanks to the same time intensity of all N_u, the jumps for different labels can be coupled through $\bar{N}$, meaning all labels $u\in I$ jump at the same time but with different jump sizes. Here, $\bar{N}$ is chosen to have compensator measure $\nu(de)dt$ with ν being uniform on $[1,2]$. (We choose the interval $[1,2]$ to avoid 0 since ν should not have mass at 0). Now $\bar{N}(dt,\varphi_u^{-1}(e-1)de)$ is a Poisson r.m. with intensity $\nu_u(de)dt$, which has the same law as N_u.

Example

Let ν_u be uniform on $[1, 2+u]$. Then $\varphi_u^{-1}(\omega) = 1 + (1+u)(\omega - 1)$, and the Assumption is satisfied.

Assumption :[Coupling of terminal condition].
We also assume that $\xi := \{\xi_u\}_{u\in I}$ can be measurably coupled,i.e. $\forall u \in I$, $\exists\bar{\xi}_u \in L^2(\bar{\mathcal{F}}_T)$ s.t. $u \mapsto \bar{\xi}_u$ is measurable and $\mathcal{L}(\xi_u) = \mathcal{L}(\bar{\xi}_u)$. We denote by $\xi \in \overline{\mathcal{M}L^2}(\mathcal{F}_T)$ if the terminal condition satisfies this assumption.

Example

Let $\xi_u := aW_T^u + \sum_{i=1}^{N_u(T)} Y_i^u$, where Y_i^u is the i-th jump of label u according to the distribution ν_u. Following the canonical coupling, we have $\bar{\xi}_u = a\bar{W}_T + \sum_{i=1}^{\bar{N}(T)} \varphi_u^{-1}(Y_i - 1)$, where $Y_i, i = 1, \ldots, \bar{N}(T)$ are i.i.d. uniform random variables on $[1, 2]$. If Assumption on intensity is satisfied, then $\bar{\xi} \in \mathcal{M}L^2(\bar{\mathcal{F}}_T)$.

The canonically coupled graphon system is now written as follows :

$$\bar{X}_u(t) = \bar{\xi}_u + \int_t^T \int_I \int_{\mathbb{R}} G(u,y) f(s,x,\bar{X}_u(s^-),\bar{Z}_u(s),\bar{\ell}_{u,s}(\cdot))\mu_{y,s}(dx)dyds$$
$$- \int_t^T \bar{Z}_u(s) d\bar{W}(s) - \int_t^T \int_E \bar{\ell}_{u,s}(\varphi_u^{-1}(e-1))\tilde{\bar{N}}(ds,de), \; u \in I, t \in [0,T]. \quad (4.2)$$

Note that $\mathcal{L}(\bar{X},\bar{Z},\bar{\ell}) = \mathcal{L}(X,Z,\ell)$.

Theorem

The coupled system (4.2) *admits a unique solution* $\bar{\Phi} := (\bar{X}, \bar{Z}, \bar{\ell}) \in \mathcal{M}$ *such that* $\bar{X} \in \mathcal{M}\mathbb{S}^2(\bar{\mathbb{F}})$. *Furthermore, the graphon mean-field BSDE system* (4.1) *also admits a unique solution* $\Phi := (X, Z, \ell) \in \mathcal{M}$, *and* $I \ni u \mapsto \mathcal{L}(X_u)$ *is measurable.*

We denote by $\mathcal{M}$ the space

$$\mathcal{M} := \{\Phi := \{(X_u, Z_u, \ell_u(\cdot)) \in \mathbb{S}^2(\mathbb{F}^u) \times \mathbb{H}^2(\mathbb{F}^u) \times \mathbb{H}^2_{\nu_u}(\mathbb{F}^u)\}_{u \in I}, \text{ such that}$$

$$\|\Phi\|_{\mathcal{M}} := \sup_{u \in I}\Big(\mathbb{E}[\sup_{t \in [0,T]} |X_u(t)|^2] + \mathbb{E}[\int_0^T |Z_u(t)|^2 dt + \mathbb{E}[\int_0^T \|\ell_{u,t}\|^2_{\nu_u} dt]\Big)^{1/2} < \infty\}.$$

For the measurability, we prove that $u \mapsto \bar{X}_u$ is measurable in u, and thus also the map $u \mapsto \mathcal{L}(\bar{X}_u)$.

Then since the canonical coupling does not change the law of the first component of solution of the original system (4.1), we have $\mathcal{L}(X_u) = \mathcal{L}(\bar{X}_u)$ for all $u \in I$. Thus $u \mapsto \mathcal{L}(X_u)$ is measurable.

Assumption A.1 (Monotonicity assumption)
$\forall u \in I, (x', x, z, \ell_1, \ell_2) \in \mathbb{R}^3 \times (L^2_{\nu_u})^2, \exists\, \phi^{x',x,z,\ell_1,\ell_2}_{u,t} \in L^2_{\nu_u}$, measurable, bounded s.t.

$$f(t, x', x, z, \ell_1) - f(t, x', x, z, \ell_2) \geq \langle \phi^{x',x,z,\ell_1,\ell_2}_{u,t}, \ell_1 - \ell_2 \rangle_{\nu_u},$$

with $\phi^{x',x,z,\ell_1,\ell_2}_{u,t}(y) \geq -1$ and $|\phi^{x',x,z,\ell_1,\ell_2}_{u,t}(y)| \leq \psi(y)$, for some $\psi \in L^2_{\nu_u}$.

Theorem (Comparison theorem)

Let $\xi^1, \xi^2 \in \overline{\mathcal{M}L^2}(\mathcal{F}_T)$ and denote (X^i, Z^i, ℓ^i) the solution of graphon mean-field BSDE (4.1) associated to (ξ^i, f^i) , $i = 1,2$. Assume

- *f_1 satisfies Assumption A.1, and f_2 is non-decreasing in x' ;*
- *For each $u \in I \setminus H$ with H a zero Lebesgue measure subset of I, $\xi^2_u \geq \xi^1_u$ a.s. and $f_2(\omega, t, x', x, z, \ell) \geq f_1(\omega, t, x', x, z, \ell)$ a.s. for all $(t, x', x, z, \ell) \in \mathbb{R}^4 \times L^2_{\nu_u}$.*

Then for all $t \in [0, T]$ and $u \in I \setminus H$, we have $X^2_u(t) \geq X^1_u(t)$ a.s..

Strict comparison thm (under Assumption A.1 with strict inequality).

Continuity of the solution with respect to the label index u

For each $u \in I$, assume

(i) $u \to \mathcal{L}(\xi_u)$ is continuous w.r.t. the $\mathcal{W}_2$ metric.

(ii) there exists a finite collection of intervals $\{I_i : i = 1, \ldots, N\}$ such that $I = \cup_i I_i$, and for each $i \in \{1, \ldots, N\}$, we have $G(u, v)$ is continuous at u for each $v \in I \setminus H_i$ for some zero Lebesgue measure set H_i.

Then for each $i \in \{1, \ldots, N\}$, the map $I_i \ni u \to \mathcal{L}(X_u)$ is continuous w.r.t. the $\mathcal{W}_{2,T}$ metric.

Stability

Convergence issues when the underlying graphon G is induced by a sequence of graphons G_n converging to G, in the sense of cut norm $\|G_n - G\|_\square \to 0$.
(Recall that $\|G\|_\square := \sup_{A,B\in\mathcal{B}(I)} \left|\int_{A\times B} G(u,v)dudv\right|$.)

- Technical assumption : Assume that for each $u \in I$, the driver f can be characterized as $\mathbb{E}^e_u[\tilde{f}(\cdot,\ell(e))]$, where $\mathbb{E}^e_u$ means taking integration of $\tilde{f}(\cdot,\ell(e))$ with respect to e under the measure ν_u, and $\tilde{f}$ is Lipschitz continuous in all parameters except time t.

Then the corresponding solution of the G_n graphon mean-field BSDE converges (in some sense) to the solution of the G graphon mean-field BSDE and the law of the X component also converges in an integral sense of the Wasserstein distance $\mathcal{W}_{2,T}$ on I :
$\int_I \mathcal{W}_{2,T}(\mathcal{L}(X_u),\mathcal{L}(X^n_u)) \to 0$.

Theorem

Let (X,Z,ℓ) and (X^n,Z^n,ℓ^n) be the solutions of (4.1) associated with graphons G and G_n, terminal condition ξ and ξ^n, respectively. Then

$$\mathbb{E}\Big[\int_I\Big(\sup_{t\in[0,T]}|X^n_u(t)-X_u(t)|^2+\int_0^T|Z^n_u(t)-Z_u(t)|^2dt+\int_0^T\|\ell^n_{u,t}-\ell_{u,t}\|^2_{\nu_u}dt\Big)du\Big]$$
$$\leq C\Big[\int_I\mathbb{E}|\xi_u-\xi^n_u|^2du+C_M\|G-G_n\|_\square+\varepsilon(M)\Big],$$

where M is some large constant, C_M is some constant depends on M and $\varepsilon(M)$ is some constant converging to 0 as M goes to ∞.
If $\|G_n-G\|_\square\to 0$ and $\mathbb{E}[\int_I|\xi_u-\xi^n_u|^2du]\to 0$ as $n\to\infty$, it follows that $\mathbb{E}\Big[\int_I\Big(\sup_{t\in[0,T]}|X^n_u(t)-X_u(t)|^2+\int_0^T|Z^n_u(t)-Z_u(t)|^2dt+\int_0^T\|\ell^n_{u,t}-\ell_{u,t}\|^2_{\nu_u}dt\Big)du\Big]\to 0$, and consequently $\int_I\mathcal{W}_{2,T}(\mathcal{L}(X_u),\mathcal{L}(X^n_u))\to 0$.

Proof : use a truncation and approximation argument

Example (Converging graphons)

For a size n adjacency matrix A, we define the associated step graphon G_A as :

$$G_A(u,v) := A_{ij}, \qquad \text{for} \quad (u,v) \in I_i^n \times I_j^n,$$

where $I_i^n := ((i-1)/n, i/n]$, for $i = 2, \ldots, n$ and $I_1^n := [0, 1/n]$.
Let ζ^n be the adjacency matrix of an Erdös-Rényi random graph $\mathcal{G}(n, p_n)$. If $p_n = p$ is fixed as $n \to \infty$, then it is well known that, as $n \to \infty$, the associated graphon G_{ζ^n} converges in cut norm to the constant graphon $G \equiv p$.

The following theorem gives another stability result which provides the convergence of graphon mean-field BSDEs in the space $\mathcal{M}$.

Theorem

$$\sup_{u\in I}\mathbb{E}\Big[\sup_{t\in[0,T]}|X_u^n(t)-X_u(t)|^2+\int_0^T|Z_u^n(t)-Z_u(t)|^2dt+\int_0^T\|\ell_{u,t}^n-\ell_{u,t}\|_{\nu_u}^2dt\Big]$$
$$\leq C\Big[\sup_{u\in I}\mathbb{E}|\xi_u-\xi_u^n|^2+C(M)\|G-G_n\|_{\infty\to\infty}+\varepsilon(M)\Big]\to 0.$$

Consequently, if $\|G_n-G\|_{\infty\to\infty}\to 0$ *and* $\sup_{u\in I}\mathbb{E}[|\xi_u-\xi_u^n|^2]\to 0$ *as* $n\to\infty$, *then*

$$\mathcal{W}_{2,T}^{\mathcal{M}}(\mathcal{L}(X),\mathcal{L}(X^n))\to 0.$$

Convergence of interacting particle systems to Graphon MF-BSDEs

Consider a sequence of N particle graphon interacting systems.

$$
\begin{aligned}
X_i^N(t) = \xi_i^N + \int_t^T \frac{1}{N}\sum_{j=1}^N \zeta_{ij}^N f(s, X_j^N(s^-), X_i^N(s^-), Z_i^N(s), \ell_s^{N,i}(\cdot))ds \\
- \int_t^T Z_i^N(s) d\widehat{W}_i(s) - \int_t^T \int_E \ell_s^{N,i}(e)\widetilde{\widehat{N}}_i(ds, de), \quad t \in [0, T]
\end{aligned} \tag{4.3}
$$

where for the i-th particle $\widehat{W}_i := W_{\frac{i}{N}}$ and $\widehat{N}_i(dt, de) = N_{\frac{i}{N}}(dt, de)$ are chosen to be the same ones associated to label i/N.

Assume $\xi_i^N \in L^2(\mathcal{F}_T^{i/N})$ for all $i = 1, \ldots, N$.

Here, $(\zeta_{ij}^N : i, j \in [N] \times [N])$ is a $N \times N$ symmetric matrix, describing the strength of interaction between particle i and j.

Define the space $\mathcal{M}^N :=$
$\{\Phi^N := \{(X_i, Z_i, \ell_i(\cdot)) \in \mathbb{S}^2(\mathbb{F}^{i/N}) \times \mathbb{H}^2(\mathbb{F}^{i/N}) \times \mathbb{H}^2_{\nu_i^N}(\mathbb{F}^{i/N})\}_{i=1}^N$, s.t.
$\|\Phi^N\|_{\mathcal{M}^N} := \max_{i=1,\dots,N}\Big(\mathbb{E}[\sup_{t\in[0,T]} |X_i(t)|^2] + \mathbb{E}[\int_0^T |Z_i(t)|^2 dt +$
$\mathbb{E}[\int_0^T \|\ell_{i,t}\|^2_{\nu_i} dt]\Big)^{1/2} < \infty\}$,
where $\nu_i^N := \nu_{i/N}$.

Theorem

The N-coupled BSDE system (4.3) *admits a unique solution* $\Phi^N \in \mathcal{M}^N$.

Assumption : For a given graphon G, we say that $\zeta^N := \{\zeta^N_{ij}\}_{i,j\in[N]}$ satisfies the regularity assumption with graphon G if either :

(i) $\zeta^N_{ij} = G(\frac{i}{N}, \frac{j}{N})$;

(ii) $\zeta^N_{ij} = \mathrm{Bernoulli}\big(G(\frac{i}{N}, \frac{j}{N})\big)$ independently for all $1 \leq i \leq j \leq N$ and independent of $\{W_u, N_u, \xi_u : u \in I\}$.

Convergence of finite particle systems to the graphon BSDE

Theorem

Suppose that ζ^N *satisfies the regularity assumption with graphon G, G is Lipschitz continuous and* $\max_{i=1,\dots,N}\mathbb{E}|\xi_i^N-\xi_{\frac{i}{N}}|^2=O(N^{-1})$. *Then the unique solution* Φ^N *of* (4.3) *converges to the unique solution of* (4.1) *with the convergence rate* $1/\sqrt{N}$ *and*

$$\max_{i=1,\dots,N}\mathbb{E}\left[\sup_{t\in[0,T]}|X_i^N(t)-X_{\frac{i}{N}}(t)|^2+\int_0^T|Z_i^N(t)-Z_{\frac{i}{N}}(t)|^2dt+\int_0^T\|\ell_t^{i,N}-\ell_t^{\frac{i}{N}}\|_\nu^2dt\right]$$
$$\leq CN^{-1}+C\max_{i=1,\dots,N}\mathbb{E}|\xi_i^N-\xi_{\frac{i}{N}}|^2=O(N^{-1}),$$

for all $N\in\mathbb{N}$ *and some constant* C. *Furthermore, for* $\kappa_t^N=\frac{1}{N}\sum_{i=1}^N\delta_{X_i^N(t)}$ *and* $\kappa_t=\int_I\mathcal{L}(X_u(t))du$, *we have*

$$\sup_{t\in[0,T]}\mathbb{E}\left[(\mathcal{W}_2(\kappa_t^N,\kappa_t))^2\right]\leq CN^{-1/2}. \tag{4.4}$$

Similar convergence result is obtained when the heterogeneous interaction weights are sampled from a sequence of convergent graphons (instead of from the limiting graphon)

Graphon dynamic risk measure

For $\xi \in \mathcal{M}L^2(\mathcal{F}_T)$ (i.e. $\xi_u \in \mathcal{F}_T^u \; \forall u \in I$, and $\xi^u, u \in I$ can be canonically coupled to satisfy the label measurability) representing a financial position at T, we interpret $\rho_{u,t}(\xi, T) := -X_u(t,\xi,T)$, for each $u \in I$, where $\{X_u(t,\xi,T)\}_{u\in I}$ is the solution of the graphon mean-field BSDE system, as the risk measure of ξ at time t and position $u \in I$. Then $\rho_t(\xi, T) := \{\rho_{u,t}(\xi, T)\}_{u\in I}$ is called the graphon associated dynamic risk measure.

Interpretation : a regulator imposes the capital to be ξ at time T, and the risk measure $\rho_t(\xi, T)$ is interpreted as the acceptable levels of liquidity at time t, for a given driver capturing how a representative bank's position evolves with dependence on the heterogeneous mean-field interactions in the system.

Properties of global dynamic risk measures

- *Consistency*. Let $\tau < T$ be a stopping time. Then $\forall t \leq \tau,\ \rho_t(\xi, T) = \rho_t(-\rho_\tau(\xi, T), \tau)$ a.s. (consequence of the uniqueness result).

- *Continuity*. Let $\{\tau^\alpha, \alpha \in \mathbb{R}\}$ be a family of stopping times converging a.s. to τ^{α_0} as $\alpha \to \alpha_0$. Let $\{\xi^\alpha, \alpha \in \mathbb{R}\}$ be a sequence of random families s.t. for each $\alpha \in \mathbb{R}$, $\xi^{\alpha,u}$ is $\mathcal{F}^u_{\tau^\alpha}$-measurable, $u \in I$ and $\mathbb{E}[\operatorname{ess\,sup}_{\alpha,u}(\xi^{\alpha,u})^2] < \infty$. Suppose also that $\xi^{\alpha,u}$ converges a.s. to a $\mathcal{F}^u_{\tau^{\alpha_0}}$-measurable r.v. ξ^u as $\alpha \to \alpha_0$. Then for each stopping time $\hat{\tau} < \tau^\alpha, \alpha \in \mathbb{R}$, the r.v. $\rho_{\hat{\tau}}(\xi^\alpha, \tau^\alpha) \to \rho_{\hat{\tau}}(\xi, \tau^{\alpha_0})$ a.s. and the processes $\rho_u(\xi^\alpha, \tau^\alpha) \to \rho_u(\xi, \tau^{\alpha_0})$ for all $u \in I$, as $\alpha \to \alpha_0$.

- *Monotonicity*. ρ is nonincreasing with respect to ξ. i.e. , for each $T > 0$ and each $\xi^1, \xi^2 \in \mathcal{M}L^2(\mathcal{F}_T)$, if $\xi^1 \geq \xi^2$ a.s., then a.s. $\rho_t(\xi^1, T) \leq \rho_t(\xi^2, T), 0 \leq t \leq T$. (consequence of comparison theorem)

- *Homogeneity :* If f is positively homogeneous wrt (x', x, z, ℓ), i.e., for $a > 0$, $f(t, ax', ax, az, al) = af(t, x', x, z, \ell)$, then the risk measure ρ is positively homogeneous wrt ξ, that is, for all $\lambda \geq 0$, $t \in [0, T]$ and $\xi \in \mathcal{M}L^2(\mathcal{F}_T)$, we have $\rho_t(\lambda\xi, T) = \lambda\rho_t(\xi, T)$.
- *Translation invariance :* If f depends only on $(t, x' - x, z, \ell)$, then the risk measure satisfies the translation invariance property : for any $\xi \in \mathcal{M}L^2(\mathcal{F}_T), t_0 \in [0, T]$ and $\xi' \in \mathcal{M}L^2(\mathcal{F}_{t_0})$,

$$\rho_t(\xi + \xi', T) = \rho_t(\xi, T) - \xi' \text{ for all } t \in [t_0, T].$$

- *No Arbitrage*. when strict comparison holds, then for each $T > 0$ and each $\xi^1, \xi^2 \in \mathcal{M}L^2(\mathcal{F}_T)$, if $\xi^1 \geq \xi^2$ and $\rho_t(\xi^1, T) = \rho_t(\xi^2, T)$ a.s. on an event $A \in \mathcal{F}_t$, then $\xi^1 = \xi^2$ a.s. on A.
- *Convexity* If f is concave with respect to (x, z, l), then the dynamic risk measure is convex, that is for any $\lambda \in [0, 1]$ and $\xi^1, \xi^2 \in \mathcal{M}L^2(\mathcal{F}_T)$, we have

$$\rho_t(\lambda\xi^1 + (1-\lambda)\xi^2, T) \leq \lambda\rho_t(\xi^1, T) + (1-\lambda)\rho_t(\xi^2, T).$$

Special case. No Graphons and f independent on x and concave Quenez-A.S. SPA 2013

$$-dX_t = f(t,\omega,Z_t,l_t(\cdot))dt - Z_t dW_t - \int_{\mathbb{R}^*} l_t(e)\tilde{N}(dt,de); \quad X_T = \xi,$$

Consider the polar function of $f(t,\omega,z,\ell)$:

$$f^*(\omega,t,\alpha^1,\alpha^2) := \sup_{(z,\ell)\in\mathbb{R}\times L^2_\nu} [f(\omega,t,z,\ell) - \alpha^1 z - \langle\alpha^2,\ell\rangle_\nu].$$

Let $\mathcal{A}_{\mathcal{T}}$ the set of predictable proc. $\alpha_s = (\alpha^1_s,\alpha^2_s)$ s.t. $f^*(t,\alpha^1,\alpha^2) \in \boldsymbol{H}^2$ (it implies in particular $\alpha^2_s(u) \geq -1$). For $\alpha \in \mathcal{A}_{\mathcal{T}}$, let Q^α be the probability (absolutely cont. wrt to P) which admits Γ^α_T as density wrt P on $\mathcal{F}_{\mathcal{T}}$, where

$$d\Gamma^\alpha_t = \Gamma^\alpha_{t-}\big(\alpha^1_t dW_t + \int_{\mathbb{R}^*} \alpha^2_t(e) d\tilde{N}(dt,de)\big); \quad \Gamma^\alpha_0 = 1.$$

Then

$$-X_0 = \sup_{\alpha\in\mathcal{A}_T} [\mathbb{E}^{Q^\alpha}(-\xi) - \zeta(\alpha,\, T)]$$

where the function ζ, called *penalty function*, is defined, by

$$\zeta(\alpha,\, T) := \mathbb{E}^{Q^\alpha}[\int_0^T f^*(s,\alpha_s)ds]$$

Example : the entropic risk measure :

$$\rho_t(\xi, T) := \frac{1}{\gamma} \ln \mathbb{E}[\exp(-\gamma\xi) \mid \mathcal{F}_t]$$

is associated to the BSDE with driver $g(z) := \frac{1}{2}\gamma z^2$
In this case the penalty function is

$$\zeta(Q) = \mathbb{E}[\frac{dQ}{dP} \ln \frac{dQ}{dP}]$$

Dual representation

$$X_u(t) = \xi_u + \int_t^T \int_I \int_{\mathbb{R}} G(u,y) f(s,x,X_u(s^-),Z_u(s),\ell_{u,s}(\cdot))\mu_{y,s}(dx)dyds$$
$$- \int_t^T Z_u(s)dW_u(s) - \int_t^T \int_E \ell_{u,s}(e)\widetilde{N}_u(ds,de), \quad \text{for } t \in [0,T],\ u \in I,$$

Suppose that f is concave with respect to (x', x, z, ℓ) and non-decreasing in x'. Let F_u denote the drift driver of the u component :

$$F_u(\omega, t, \mathcal{L}(X_t), x, z, \ell(\cdot)) := \int_I \int_{\mathbb{R}} G(u,y) f(t, x', x, z, \ell(\cdot))\mu_{y,t}(dx')dy.$$

For each (ω, t) and $u \in I$, let $(F_u)^*$ the Fenchel-Legendre transform defined as

$$(F_u)^*(\omega, t, \mathcal{L}(Y), \beta_u, \alpha_u^1, \alpha_u^2) := \sup_{(X,x,z,\ell)\in L^{2,I}(\bar{\mathcal{F}}_t)\otimes\mathbb{R}^2\otimes L^2_{\nu_u}} \{F_u(\omega, t, \mathcal{L}(X), x, z, \ell)$$
$$- \langle X, Y\rangle_{L^{2,I}} - \beta_u x - \alpha_u^1 z - \langle \alpha_u^2, \ell_u\rangle_{\nu_u}\}.$$

For $u \in I$ and given predictable process $\alpha_u = (\alpha_u^1, \alpha_u^2)$, let Q_u^α be the proba abs. continuous wrt $\mathbb{P}$, which admits $\Gamma^{\alpha_{u,T}}$ as density, where Γ^{α_u} is solution of

$$d\Gamma^{\alpha_{u,t}} = \Gamma^{\alpha_{u,t-}}\Big(\alpha_{u,t}^1 dW_u(t) + \int_E \alpha_{u,t}^2(e) d\widetilde{N}_u(dt,de)\Big), \quad \Gamma^{\alpha_{u,0}} = 1.$$

Let $\mathcal{A}_T^I$: set of families of processes $(\gamma_t, \beta_t, \alpha_t)_{t\in[0,T]}$, where $(\gamma_t)_{t\in[0,T]}$ (with $\gamma_t := (\gamma_t^{u,v})_{u,v\in I}$) progressively measurable, $(\beta_t, \alpha_t)_{t\in[0,T]}$ predictable, and s.t.

- $\alpha := \{\alpha_u\}_{u\in I}$ s.t. $\forall u \in I$, $\int_0^T (\alpha_{u,s}^1)^2 ds + \int_0^T \|\alpha_{u,s}^2\|_{\nu_u}^2 ds$ is bounded, and $\alpha_{u,t}^2(y) > -1$ $\nu_u(dy)$-a.s. for all $t \in [0,T]$.
  (This implies $\Gamma^{\alpha_{u,t}} > 0$ a.s. on $[0,T]$ and $(\Gamma^{\alpha_{u,t}})_{t\in[0,T]} \in \mathbb{S}^2$).
- $\forall (u,v) \in I$, $(\Gamma_{v,t}^\alpha e^{\int_0^t \gamma_y^{u,v} dy})_{t\in[0,T]} \in \mathbb{H}^2$;
- $\forall v \in I$, $\{(F_v)^*\Big(t, \Big(\frac{\Gamma_t^{\alpha_{v_1}} H_{0,t}^{\beta_{v_1},\gamma^{v,v_1}} \gamma_t^{v,v_1}}{\mathbb{E}[\Gamma_t^{\alpha_v} \int_I H_{0,t}^{\beta_v,\gamma^{v_1,v}} dv_1]}\Big)_{v_1\in I}, \beta_{v,t}, \alpha_{v,t}^1, \alpha_{v,t}^2(\cdot)\Big)\}_{t\in[0,T]} \in \mathbb{H}^2$.

Theorem (Dual representation)

For each $t \in [0,T]$, we have :

$$\mathbb{E}[\int_I \rho_{v,t}(\xi,T)dv] =$$

$$\sup_{(\gamma,\beta,\alpha)\in\mathcal{A}_T^I} \{\int_I \mathbb{E}^{Q_v^\alpha}[-(\int_I H_{t,T}^{\beta_v,\gamma^{u,v}} du)\xi_v]dv - \int_I \zeta_{v,t}(\gamma,\beta,\alpha,T)dv\},$$

where $\zeta_{v,t}(\gamma,\beta,\alpha,T) :=$

$$\int_t^T \mathbb{E}^{Q_v^\alpha}\Big[(\int_I H_{t,s}^{\beta_v,\gamma^{u,v}} du)(F_v)^*\Big(s, \Big(\frac{\Gamma_s^{\alpha_{v_1}} H_{t,s}^{\beta_{v_1},\gamma^{v,v_1}} \gamma_s^{v,v_1}}{\mathbb{E}[\Gamma_s^{\alpha_v}\int_I H_{t,s}^{\beta_v,\gamma^{v_1,v}} dv_1]}\Big)_{v_1}, \beta_{v,s}, \alpha_{v,s}^1, \alpha_{v,s}^2(\cdot)\Big)\Big] ds,$$

$$H_{t,s}^{\beta,\gamma} := \exp\{\int_t^s (\beta_y + \gamma_y) dy\}.$$

Moreover, $\exists(\bar\gamma,\bar\beta,\bar\alpha) \in \mathcal{A}_T^I$ *attaining the supremum*

Steps of the proof :

- establish bounds on the effective domain of F^*.
- provide some explicit form to conjugacy relations relying on a Mean-Field Graphon FSDE

Happy birthday Ying !

Wasserstein distances. Given a Polish space $\mathcal{S}$, denote by $\mathcal{D}([0,T],\mathcal{S})$ the space of RCLL functions from $[0,T]$ to $\mathcal{S}$, equipped with the Skorokhod topology. Let $\mathcal{D}_m := \mathcal{D}([0,T],\mathbb{R}^m)$. Denote by $\mathcal{P}(\mathcal{S})$ the space of probability measures on $\mathcal{S}$.

Wasserstein distances between two probability measures μ and ν :

$$\mathcal{W}_2(\mu,\nu) := \left(\inf\{\mathbb{E}[|X_1 - X_2|^2] : \mathcal{L}(X_1)=\mu, \mathcal{L}(X_2)=\nu\}\right)^{1/2}, \quad \text{for} \quad \mu,\nu \in \mathcal{P}(\mathbb{R}^m),$$

$$\mathcal{W}_{2,T}(\mu,\nu) := \left(\inf\{\sup_{t\in[0,T]} \mathbb{E}|X_1(t) - X_2(t)|^2 : \mathcal{L}(X_1)=\mu, \mathcal{L}(X_2)=\nu\}\right)^{1/2}, \quad \text{for} \quad \mu,$$

For two families of probability measures $\mu = \{\mu_u\}_{u\in I}$ and $\nu = \{\nu_u\}_{u\in I}$, set

$$\mathcal{W}_2^{\mathcal{M}}(\mu,\nu) := \sup_{u\in I} \mathcal{W}_2(\mu_u,\nu_u), \text{ for } \mu,\nu \in \mathcal{P}(\mathcal{M}L^2) \text{ for all } t\in[0,T],$$

and

$$\mathcal{W}_{2,T}^{\mathcal{M}}(\mu,\nu) := \sup_{u\in I} W_{2,T}(\mu_u,\nu_u), \text{ for } \mu,\nu \in \mathcal{P}(\mathcal{M}\mathbb{S}^2).$$

For Lipschitz continuity, we need a stronger assumption.
There exists a finite collection of intervals $\{I_i : i = 1, \ldots, N\}$ such that $I = \cup_i I_i$, and for some constant C, we have for all $u_1, u_2 \in I_i$, $v_1, v_2 \in I_j$, and $i, j \in \{1, \ldots, N\}$,

$$\mathcal{W}_2(\mathcal{L}(\xi_{u_1}), \mathcal{L}(\xi_{u_2})) \leq C|u_1 - u_2|,$$

and,

$$|G(u_1, v_1) - G(u_2, v_2)| \leq C(|u_1 - u_2| + |v_1 - v_2|).$$